ARKIV FÖR FYSIK Bd 3 nr 1

Communicated 7 June 1950 by Nils Zeilon and Ivar Waller

On determination of proton-proton interaction from scattering experiments

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With 2 figures in the text

1. Introduction

It is a well-known fact that small energy proton-proton scattering experiments can be well explained by several different potential functions. Usually the well potential, the exponential potential and the Yukawa and Gauss potentials have been taken into account (cf. e.g. ref. [5, 19]). On the other hand it can be proved that scattering data under certain conditions determine the potential function uniquely. Therefore it seems to be of considerable interest to make a direct attempt of calculating the proton-proton interaction from available experiments. In some previous papers [9, 10, 11] the problem of determining a central interaction from the asymptotic phase has been solved in principle. Later Bargmann [3, 4] gave several examples of phase-equivalent potentials, but this ambiguity was settled by Levinson's uniqueness theorem [21, 22]. In these treatments, however, Coulomb forces were excluded. It is the aim of the present paper to investigate this case; the corresponding uniqueness theorem has been constructed by Jauho [20].

There are some great advantages in studying scattering between identical particles. From Pauli's exclusion principle it follows that there can be no ^3S-scattering. As a matter of fact it turns out that for low energies (<10 MeV) the ^1S-state is completely predominating.

2. The Coulomb wave equation; orthogonality theorems

The radial equation for pure Coulomb scattering can be written

$$\frac{d^2 u}{d r^2} + \left[k^2 - \frac{2 \alpha k}{r} - \frac{l(l+1)}{r^2}\right] \cdot u = 0 \tag{1}$$

with $k^2 = \frac{M \varepsilon}{\hbar^2}$ and $\alpha = \frac{M e^2}{2 \hbar^2 k}$, where M is the proton mass and ε the energy in the laboratory system.

Now we choose two solutions u and v, forming a fundamental system, in the following way: $u(0) = 0$, $u_{r \to \infty} \sim \sin(kr + \eta)$. It turns out that

$$\eta = -\alpha \log 2kr - \tfrac{1}{2} l \pi + \arg \Gamma(i\alpha + l + 1)$$

(cf. [11] and [28]). The other function v is defined by $v_{r \to \infty} \sim \cos(kr + \eta)$ with the same η as in u.

In the following treatment we need some orthogonality theorems. Then we have to consider two energy values, i.e. two values of k, say k and $\varkappa$. Observing that $k\alpha_k = \varkappa \alpha_\varkappa$ we easily obtain

$$u_\varkappa u_k'' - u_k u_\varkappa'' + (k^2 - \varkappa^2) u_k u_\varkappa = 0. \tag{2}$$

Integrating from zero to infinity and making use of the asymptotic expression for u we get

$$\int_0^\infty u_k u_\varkappa \, dr = \frac{\pi}{2} \delta(k - \varkappa). \tag{3}$$

If we start from the equation

$$u'' + f(r) u = 0 \tag{4}$$

with the fundamental system (u, v) we can form a third order differential equation with the solutions $U = u^2$, $V = uv$ and $W = v^2$. Taking V for example we have

$$\begin{aligned} V' &= u'v + uv' \\ V'' &= u''v + 2u'v' + uv'' \\ V''' &= u'''v + 3u''v' + 3u'v'' + uv'''. \end{aligned}$$

On the other hand we obtain by differentiating equation (4)

$$u''' + f(r) u' + f'(r) u = 0$$

$$v''' + f(r) v' + f'(r) v = 0.$$

Thus $V''' = -f(r) \cdot (u'v + uv') - 2f'(r) uv - 3f(r)(u'v + uv')$ and

$$V''' + 4f(r) V' + 2f'(r) V = 0. \tag{5}$$

We also need a second order equation in U:

$$U = u^2, \quad U' = 2uu' \text{ and } U'' = 2uu'' + 2u'^2.$$

Thus $U'' - 4uu'' + (2uu'' - 2u'^2) = 0$ or

$$U'' + 4f(r)U + (2uu'' - 2u'^2) = 0. \tag{6}$$

Now we can directly write down the following system (cf. [10, 18])

$$\left.\begin{aligned} &U_k'' + 4\left[k^2 - \frac{l(l+1)}{r^2} - \frac{2k\alpha_k}{r}\right]U_k + (2u_k u_k'' - 2u_k'^2) = 0 \\ &V_\varkappa''' + 4\left[\varkappa^2 - \frac{l(l+1)}{r^2} - \frac{2\varkappa\alpha_\varkappa}{r}\right]V_\varkappa' + 4\left[\frac{l(l+1)}{r^3} + \frac{\varkappa\alpha_\varkappa}{r^2}\right]V_\varkappa = 0 \end{aligned}\right\} \tag{7}$$

Multiplying these equations by $V_\varkappa'$ and $-U_k$ and integrating from zero to infinity we get

$$[U_k' V_\varkappa' - U_k V_\varkappa'']_{r\to\infty} + 4(k^2 - \varkappa^2)\int_0^\infty U_k V_\varkappa' \, dr - \int_0^\infty 4 U_k V_\varkappa \left(\frac{l(l+1)}{r^3} + \frac{\varkappa\alpha_\varkappa}{r^2}\right) dr$$

$$+ \left[V_\varkappa \cdot (2u_k u_k'' - 2u_k'^2)\right]_0^\infty - \int_0^\infty V_\varkappa \cdot (2u_k u_k''' - 2u_k' u_k'') \, dr = 0.$$

As is easily found $u_k u_k''' - u_k' u_k'' = -2\left(\frac{l(l+1)}{r^3} + \frac{k\alpha_k}{r^2}\right) \cdot U_k$, and after some trivial calculations, where we make use of the asymptotic behaviour of u, U and V we obtain

$$\int_0^\infty \frac{1}{\varkappa} U_k(r) V_\varkappa'(r) \, dr = -\frac{\pi}{8}\delta(k - \varkappa) \tag{8}$$

$$\int_0^\infty \frac{1}{\varkappa} U_k'(r) V_\varkappa(r) \, dr = \frac{\pi}{8}\delta(k - \varkappa). \tag{9}$$

3. The complete wave equation and the nuclear phase

We now turn to the case in which there exists a central nuclear interaction besides the Coulomb force. The corresponding radial equation reads

$$\frac{d^2 y}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2\alpha k}{r}\right] y = -V(r) y. \tag{10}$$

The regular solution has the asymptotic form

$$y \sim \sin(kr + \eta + \delta) \tag{11}$$

where $\delta = \delta(k)$ is the nuclear phase. As in [9] and [17] we find the exact equation

$$k \sin \delta(k) = \int_0^\infty V(r) \cdot u_k(r) \cdot y_k(r)\, dr. \tag{12}$$

4. Different possibilities for obtaining approximate solutions

Our problem can now be formulated: Given the phase function $\delta(k)$ for every positive k, find the potential function $V(r)$. We will try to tackle this problem starting from equation (12). The main difficulty is, of course, that the unknown function $V(r)$ is involved implicitly also in y_k. We will try to get around this complication by replacing y_k by some suitable easy-handled function. The most natural choice seems to be

$$y_k \rightarrow a u_k + b v_k$$

or with respect to the asymptotic behaviour

$$y_k \rightarrow u_k \cdot \cos \delta(k) + v_k \cdot \sin \delta(k).$$

Unfortunately this relation is not compatible with the boundary condition $y_k(0) = 0$ unless $\delta(k) = 0$. Anyhow, let us suppose that the phase shift is small and replace y_k by u_k. This agreed upon we get

$$k \sin \delta(k) = \int_0^\infty V(r) \cdot U_k(r) \cdot dr \tag{13}$$

which corresponds to Born's approximation.

Another possibility is offered by the following procedure.[1] We replace y_k by $(y_k)_{k=0} = y_0$ which is independent of k. Then the equation

$$k \sin \delta(k) = \int_0^\infty V(r)\, u_k(r)\, y_0(r)\, dr$$

can be solved by using eq. (3):

$$V(r) \cdot y_0(r) = \frac{2}{\pi} \int_0^\infty k \sin \delta(k) \cdot u_k(r)\, dk.$$

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[1] I am indebted to Dr. NILS SVARTHOLM for this suggestion.

On the other hand y_0 must be a solution of

$$y_0'' - \frac{\varkappa}{r}\, y_0 = -\, y_0\, V(r),$$

the right hand side now being a known function. Solving for y_0 we obtain directly the function $V(r)$ desired.

In order to get an idea of how this process works, consider the following example

$$\sin \delta(k) = \frac{b}{4} \cdot \frac{\log\left(1 + \frac{4k^2}{\varkappa^2}\right)}{\frac{k}{\varkappa}}$$

(this corresponds to the Born approximation of $V(r) = b\varkappa \frac{e^{-\varkappa r}}{r}$ without Coulomb forces). Then we must replace u_k by $\sin kr$:

$$V(r) \cdot y_0 = \frac{2}{\pi} \cdot \frac{b\varkappa}{4} \int\limits_0^\infty \sin kr \cdot \log\left(1 + \frac{4k^2}{\varkappa^2}\right) dk.$$

Putting $\frac{2k}{\varkappa} = \alpha$ and $\varkappa r = \lambda$ and further adding a convergence factor we obtain

$$V(r)\, y_0 = \frac{b\varkappa}{\pi r} \int\limits_0^\infty \frac{\alpha}{1+\alpha^2} \cos \frac{\lambda \alpha}{2} \cdot d\alpha = -\frac{b\varkappa}{2\pi r} \cdot \left[e^{\frac{\lambda}{2}} \cdot Ei\left(-\frac{\lambda}{2}\right) + e^{-\frac{\lambda}{2}} Ei\left(\frac{\lambda}{2}\right)\right].$$

Now y_0 shall be determined from

$$\frac{d^2 y_0}{dr^2} = -\, y_0\, V(r).$$

Putting $\lambda = 2x$ and consequently $r = \frac{2x}{\varkappa}$ we get

$$\frac{d^2 y_0}{dx^2} = \frac{b}{\pi}\left[\frac{e^x}{x} Ei(-x) + \frac{e^{-x}}{x} Ei(x)\right].$$

Thus

$$\frac{dy_0}{dx} = \frac{b}{\pi} Ei(x)\, Ei(-x) + c_1$$

and

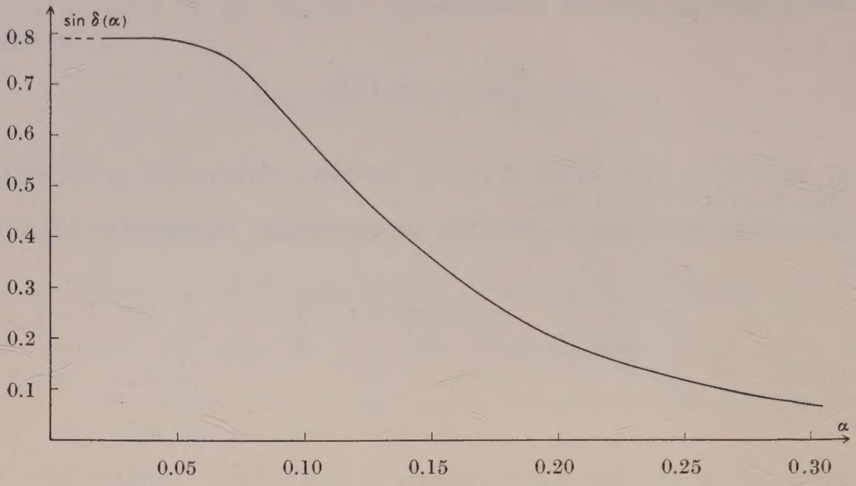

Fig. 1.

$$y_0 = \frac{b}{\pi}[x\,Ei(x)\,Ei(-x) - e^x\,Ei(-x) + e^{-x}\,Ei(x)] + c_1 x + c_2$$

Choosing $c_1 = c_2 = 0$ we obtain

$$V = -\frac{\varkappa^2}{4}\cdot\frac{e^x\,Ei(-x) + e^{-x}\,Ei(x)}{x[x\,Ei(x)\,Ei(-x) - e^x\,Ei(-x) + e^{-x}\,Ei(x)]}$$

with $x = \frac{1}{2}\varkappa r$. When x is small we have

$V \sim \frac{\varkappa^2}{2}\cdot\frac{1}{x^2 \log\frac{1}{x}}$ and for large values of x

$V \sim -\frac{\varkappa^2}{2}\cdot\frac{1}{x^2}$. Neither of these expressions bears any strong resemblance to the Yukawa potential function, and we must conclude that this approximation method is unadequate, at least for large energies. The discrepancy for small energies can be explained from the fact that the Born approximation for the phase shift is poor in this case.

We thus return to equation (13). It can easily be solved by means of the orthogonality condition (8):

$$V(r) = -\frac{8}{\pi}\int_0^\infty \sin\delta(k)\cdot V_k'(r)\cdot dk. \qquad (14)$$

We make some simple substitutions

$$k r = \varrho, \; 2\alpha = \frac{\varkappa}{k}, \; \varkappa = \frac{M e^2}{\hbar^2} \text{ and } \varkappa r = \lambda.$$

Then we have

$$V(r) = -\frac{2}{\pi} \varkappa^2 \cdot \int_0^\infty \sin \delta(\alpha) \left\{ \frac{d}{d\varrho} [u(\alpha, \varrho) \cdot v(\alpha, \varrho)] \right\}_{\varrho = \frac{\varkappa r}{2\alpha}} \cdot \frac{d\alpha}{\alpha^3} \tag{15}$$

or

$$V(r) = -\frac{2}{\pi} \varkappa^2 \cdot \int_0^\infty \frac{\varphi(\alpha, \lambda) \sin \delta(\alpha)}{\alpha^3} d\alpha \tag{16}$$

where

$$\varphi(\alpha, \lambda) = \left\{ \frac{d}{d\varrho} [u(\alpha, \varrho) v(\alpha, \varrho)] \right\}_{\varrho = \frac{\lambda}{2\alpha}}. \tag{17}$$

As is well-known the S-phase is predominating and the function $\sin \delta(\alpha)$ for $l = 0$ (see Fig. 1) is known from experiments (cf. e.g. [19] and [30] which also contain a more detailed list of references). Thus our problem has been reduced to determination of the function $\varphi(\alpha, \lambda)$. The regular function u and its derivative have been taken from the tables computed by Math. Tables Project, New York [25], while no tables of the irregular function v and its derivative have been available. For this reason rather extensive calculations of these functions have been performed, mainly for small values of α and ϱ.

5. Computation of the irregular Coulomb *S*-wave functions

When ϱ is large compared with α we can use the asymptotic expression $v \sim \cos(\varrho - \alpha \log 2\varrho + \arg \Gamma(i\alpha + 1))$. As a matter of fact it is even possible to derive a better approximation in the form of a series expansion. Putting

$$\begin{aligned} v = {} & \cos\theta \cdot [1 + \alpha \varphi_1(\varrho) + \alpha^2 \varphi_2(\varrho) + \cdots] - \\ & - \sin\theta \cdot [\alpha \psi_1(\varrho) + \alpha^2 \psi_2(\varrho) + \cdots] \end{aligned}$$

with

$$\theta = \varrho - \alpha \log 2\varrho + \arg \Gamma(i\alpha + 1)$$

and using the differential equation for v we obtain, collecting the first order terms in α:

$$\begin{cases} 2\psi_1' - \varphi_1'' = 0 \\ \dfrac{1}{\varrho^2} + 2\varphi_1' + \psi_1'' = 0. \end{cases}$$

Claiming $\varphi_1(\infty) = \psi_1(\infty) = 0$ and eliminating we get:

$$\begin{cases} \varphi_1'' + 4\,\varphi_1 = \dfrac{2}{\varrho} \\ \psi_1'' + 4\,\psi_1 = -\dfrac{1}{\varrho^2} \end{cases}$$

Making use of the well-known fundamental solution $\sin|x-\xi|$ of the equation $y''+y=0$ we easily find

$$\begin{cases} \varphi_1 = \sin 2\,\varrho\, Ci\, 2\,\varrho + \cos 2\,\varrho \left(\dfrac{\pi}{2} - Si\, 2\,\varrho\right) \\ \psi_1 = \cos 2\,\varrho\, Ci\, 2\,\varrho - \sin 2\,\varrho \left(\dfrac{\pi}{2} - Si\, 2\,\varrho\right) \end{cases}. \tag{18}$$

In a similar way collecting second order terms in α we obtain equations for φ_2 and ψ_2, and only technical difficulties prevent us from going further in this manner.

As has been pointed out by ABRAMOWITZ ([2], p. 78–79) it is possible to obtain a semi-convergent series expansion by elementary methods. Putting

$$\begin{cases} u = \sin\theta \left[1 + \dfrac{A_1}{\varrho} + \dfrac{A_2}{\varrho^2} + \cdots\right] + \cos\theta \left[\dfrac{B_1}{\varrho} + \dfrac{B_2}{\varrho_2} + \cdots\right] \\ v = \cos\theta \left[1 + \dfrac{A_1}{\varrho} + \dfrac{A_2}{\varrho^2} + \cdots\right] - \sin\theta \left[\dfrac{B_1}{\varrho} + \dfrac{B_2}{\varrho^2} + \cdots\right] \end{cases}$$

we get

$$\begin{cases} 2\,k\,A_k = \alpha\,(2\,k-1)\,A_{k-1} - [\alpha^2 - k\,(k-1)]\,B_{k-1} \\ 2\,k\,B_k = \alpha\,(2\,k-1)\,B_{k-1} + [\alpha^2 - k\,(k-1)]\,B_{k-1} \end{cases}$$

with $A_0 = 1$ and $B_0 = 0$. Solving for the coefficients yields

$$A_1 = \frac{\alpha}{2};\quad A_2 = \frac{\alpha^2}{8}\,(5-\alpha^2);\quad A_3 = \frac{\alpha}{16}\,(-4 + 17\,\alpha^2 - 3\,\alpha^4);$$

$$B_1 = \frac{\alpha^2}{2};\quad B_2 = \frac{\alpha}{4}\,(-1 + 2\,\alpha^2);\quad B_3 = \frac{\alpha^2}{48}\,(-40 + 31\,\alpha^2 - \alpha^4);$$

$$A_4 = \frac{\alpha^2}{384}\,(-564 + 769\,\alpha^2 - 106\,\alpha^4 + \alpha^6)\cdots$$

$$B_4 = \frac{\alpha}{48}\,(18 - 113\,\alpha^2 + 47\,\alpha^4 - 2\,\alpha^6)\cdots$$

Combining all terms proportional to α we find exactly the semi-convergent series of the functions φ_1 and ψ_1 defined above. After some computations we obtain the following formulae

$$
\begin{cases}
u = \sin\theta \cdot \left[1 + \alpha\varphi_1 + \alpha^2\left(\dfrac{5}{8\varrho^2} - \dfrac{141}{96\varrho^4} + \cdots\right) + \cdots\right] + \\
\quad + \cos\theta \cdot \left[\alpha\psi_1 + \alpha^2\left(\dfrac{1}{2\varrho} - \dfrac{5}{6\varrho^3} + \cdots\right) + \cdots\right] \\
u' = \cos\theta \cdot \left[1 - \alpha\varphi_1 + \alpha^2\left(-\dfrac{3}{8\varrho^2} + \dfrac{41}{32\varrho^4} - \cdots\right) + \cdots\right] + \\
\quad + \sin\theta \cdot \left[\alpha\psi_1 + \alpha^2\left(-\dfrac{1}{2\varrho} - \dfrac{1}{6\varrho^3} + \cdots\right) + \cdots\right]
\end{cases}
$$

$$
\begin{cases}
v = \cos\theta \cdot \left[1 + \alpha\varphi_1 + \alpha^2\left(\dfrac{5}{8\varrho^2} - \dfrac{141}{96\varrho^4} + \cdots\right) + \cdots\right] - \\
\quad - \sin\theta \cdot \left[\alpha\psi_1 + \alpha^2\left(\dfrac{1}{2\varrho} - \dfrac{5}{6\varrho^3} + \cdots\right) + \cdots\right] \\
v' = -\sin\theta \cdot \left[1 - \alpha\psi_1 + \alpha^2\left(-\dfrac{3}{8\varrho^2} + \dfrac{41}{32\varrho^4} - \cdots\right) + \cdots\right] + \\
\quad + \cos\theta \cdot \left[\alpha\psi_1 + \alpha^2\left(-\dfrac{1}{2\varrho} - \dfrac{1}{6\varrho^3} + \cdots\right) + \cdots\right]
\end{cases}
\tag{19}
$$

where φ_1 and ψ_1 are defined by (18).

It is interesting to observe that

$$
\begin{aligned}
u' + v &= 2\cos\theta + O(\alpha^2) \\
u - v' &= 2\sin\theta + O(\alpha^2)
\end{aligned}
$$

and consequently $u^2 + u'^2 + v^2 + v'^2 = 2 + O(\alpha^2)$ (since $u'v - uv' \equiv 1$).

For completeness the corresponding formulae will also be given in the case $l > 0$. Consider the equation

$$
u'' + \left(1 - \frac{2\alpha}{\varrho} - \frac{l(l+1)}{\varrho^2}\right) u = 0
$$

with the solutions u and v. When $\alpha = 0$ we have the fundamental system u_0, v_0 where

$$
\begin{cases}
u_0(\varrho) = \sin\left(\varrho - \dfrac{l}{2}\pi\right) \cdot \left[1 - \dfrac{(l-1)\,l\,(l+1)\,(l+2)}{8\varrho^2} + \cdots\right] + \\
\quad + \cos\left(\varrho - \dfrac{l}{2}\pi\right) \cdot \left[\dfrac{l(l+1)}{2\varrho} + \cdots\right]
\end{cases}
$$

$$
\begin{cases}
v_0(\varrho) = \cos\left(\varrho - \dfrac{l}{2}\pi\right) \cdot \left[1 - \dfrac{(l-1)\,l\,(l+1)\,(l+2)}{8\varrho^2} + \cdots\right] - \\
\quad - \sin\left(\varrho - \dfrac{l}{2}\pi\right) \cdot \left[\dfrac{l(l+1)}{2\varrho} + \cdots\right]
\end{cases}
$$

and the expressions within brackets contain only a finite number of terms.

Putting

$$\theta = \varrho - \alpha \log 2\varrho + \arg \Gamma(i\alpha + l + 1)$$

we will try to find a series expansion for u and v in terms of α:

$$\begin{cases} u = u_0(\theta) \cdot [1 + \alpha\varphi_1(\varrho) + \cdots] + v_0(\theta) \cdot [\alpha\psi_1(\varrho) + \cdots] \\ v = v_0(\theta) \cdot [1 + \alpha\varphi_1(\varrho) + \cdots] - u_0(\theta) \cdot [\alpha\psi_1(\varrho) + \cdots] \end{cases}$$

where $\varphi_1, \varphi_2, \ldots$ and $\psi_1, \psi_2, \ldots$ vanish at infinity. Obviously u and v go over into u_0 and v_0 when $\alpha = 0$ and further $\begin{pmatrix} u \\ v \end{pmatrix}_{\varrho\to\infty} \sim \begin{matrix}\sin \\ \cos\end{matrix}\left(\theta - \frac{l}{2}\pi\right)$ as it should be. We limit ourselves to the determination of the functions φ_1 and ψ_1. Inserting into the differential equation we find after some calculations, comparing terms of the same order:

$$\begin{cases} \varphi_1'' - 2\psi_1' - \dfrac{2l(l+1)}{\varrho^3} = 0 \\ \psi_1'' + 2\varphi_1' + \dfrac{1}{\varrho^2} = 0. \end{cases}$$

Eliminating and using the boundary conditions we obtain

$$\begin{cases} \varphi_1'' + 4\varphi_1 = \dfrac{2}{\varrho} + \dfrac{2l(l+1)}{\varrho^3} \\ \psi_1'' + 4\psi_1 = \dfrac{2l^2 + 2l - 1}{\varrho^2} \end{cases}$$

with the solutions

$$\varphi_1 = \frac{l(l+1)}{\varrho} - (2l^2 + 2l - 1) \cdot \left[\cos 2\varrho \left(\frac{\pi}{2} - \operatorname{Si} 2\varrho\right) + \sin 2\varrho \operatorname{Ci} 2\varrho\right]$$

$$\psi_1 = (2l^2 + 2l - 1) \cdot \left[\sin 2\varrho \left(\frac{\pi}{2} - \operatorname{Si} 2\varrho\right) - \cos 2\varrho \operatorname{Ci} 2\varrho\right].$$

Thus

$$\begin{cases} u = u_0(\theta) \cdot [1 + \alpha\varphi_1] + v_0(\theta) \cdot \alpha\psi_1 + O(\alpha^2) \\ v = v_0(\theta) \cdot [1 + \alpha\varphi_1] - u_0(\theta) \cdot \alpha\psi_1 + O(\alpha^2) \end{cases}$$

We also easily find the relations

$$u' = v_0(\theta) \cdot [1 - \alpha\varphi_1] + u_0(\theta) \cdot \alpha\psi_1 + O\left(\frac{1}{\varrho^2}\right) + O(\alpha^2)$$

$$v' = -u_0(\theta) \cdot [1 - \alpha\varphi_1] + v_0(\theta) \cdot \alpha\psi_1 + O\left(\frac{1}{\varrho^2}\right) + O(\alpha^2).$$

For illustration we choose $\alpha = 0.01$, $\varrho = 6$ and $\alpha = 0.02$, $\varrho = 3$. From (19) we get $v = 0.951927$ and $v = -0.985270$, while direct computation (Table I) yields $v = 0.951932$ and $v = -0.985269$ respectively.

For moderate values of ϱ and α we can use the integral formulae for v and v' given in a previous paper [11].[1] With the notations

$$\lambda = \varkappa r,\ \gamma = \frac{1}{\varrho} \log \frac{1}{x},\ \varphi = \varrho x - \alpha \log \frac{1+x}{1-x},$$

$\omega = \operatorname{arctg} \gamma$ and $A = \sqrt{\dfrac{1 - e^{-2\alpha\pi}}{2\alpha\pi}}$ we have:

$$\left\{ \begin{aligned} v &= A \int_0^1 (e^{2\alpha\omega} - \varrho \sin \varphi)\, dx \\ v' &= -A \int_0^1 \left(\frac{2\alpha}{\varrho} \frac{\gamma}{1+\gamma^2} e^{2\alpha\omega} + \varrho x \cos \varphi + \sin \varphi \right) dx. \end{aligned} \right. \tag{20}$$

These formulae are, of course, impractical for computation of tables but nevertheless useful for checking purposes.

For small values of ϱ the power series expansions give very accurate values

$$\left\{ \begin{aligned} v &= \sqrt{\frac{e^{2\pi\alpha} - 1}{2\pi\alpha}} \cdot \{a_0 + a_1 \varrho + a_2 \varrho^2 + \cdots + \log \varrho\, (b_1 \varrho + b_2 \varrho^2 + \cdots)\} \\ v' &= \sqrt{\frac{e^{2\pi\alpha} - 1}{2\pi\alpha}} \cdot \{a_0' + a_1' \varrho + a_2' \varrho^2 + \cdots + \log \varrho\, (b_0' + b_1' \varrho + \cdots)\} \end{aligned} \right. \tag{21}$$

The coefficients are obtained from the system

$$\left\{ \begin{aligned} & k\,(k+1)\, a_{k+1} + (2\,k+1)\, b_{k+1} - 2\,\alpha\, a_k + a_{k-1} = 0 \\ & k\,(k+1)\, b_{k+1} - 2\,\alpha\, b_k + b_{k-1} = 0 \\ & a_{k-1}' = k\, a_k + b_k,\ b_{k-1}' = k\, b_k,\ b_1 = 2\,\alpha\, a_0. \end{aligned} \right.$$

If we choose $a_0 = 1$ and $a_1 = 2\,\alpha\, c$, where

$$c = \log 2 + 2\gamma - 1 + \text{R.P.} \frac{\Gamma'(i\,\alpha)}{\Gamma(i\,\alpha)} \quad (\gamma = \text{Euler's constant})$$

we obtain the function v desired [28]. Thus we know the behaviour of v for small values of ϱ and for large values of ϱ. In order to fill the gap between these extremes we must turn to numerical integration. Since the derivative v'

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[1] The corresponding formulae for the regular functions u and u' were first given in [23].

is logarithmically divergent at the origin the integration must start at some $\varrho > 0$ (in practice $\varrho = 0.05$). The calculations were performed on the Swedish automatic relay computing machine BARK in Stockholm. During this work it soon appeared that the checking plays a fundamental rôle. Three methods (a difference method, Runge-Kutta's method and Taylor series expansion) were tried, all of which worked successfully, but the last method (suggested by KJELLBERG) turned out to be superior to the other ones. Checking could be accomplished by back-ward integration of every step and comparation with the old values. When the difference was $\geq 2^{-24}$ the interval was diminished. Larger errors were usually due to occasional failures by the machine and in such cases a new run of that particular step was undertaken.

Moreover some values were also checked by the formulae (20), especially for large values of ϱ. Further a very efficient mathematical check, viz.

$$u' v - u v' = 1 \tag{22}$$

has found extensive employment. As a matter of fact this identity also allows for correcting a wrong solution. Suppose that we have obtained a solution $w = \alpha u + \beta v$ (α and β unknown constants) differing by a small quantity from the wanted solution v. Then $u' w - u w'$ takes a value γ instead of unity:

$$u' (\alpha u + \beta v) - u (\alpha u' + \beta v') = \gamma$$

and hence, using (22) we get $\beta = \gamma$.

Now choose ϱ near a point where $w = 0$. In such a point we have

$$u' \sim 0, \;\; v \sim 0;$$

and

$$u \sim \pm 1, \;\; v' \sim \mp 1.$$

In the expression

$$w' = \alpha u' + \beta v'$$

both α and u' are small and we have approximately

$$v' \sim \frac{w'}{\beta}$$

(valid in this special point). From (22) we can also determine v and then α can be calculated from $w = \alpha u + \beta v$. The process can be repeated to give a better value for α. Knowing the values of α and β we can easily adjust the function w to the desired solution v.

Checks by means of (22) were undertaken at random for all values of α. The results suggest between five and six correct figures, but in many cases the accuracy is without doubt still better. For this reason it seems justified to give six figures in the table. (The numerical integration was performed with 24 binary digits, i.e. slightly more than 7 decimal digits.) For not too small values of ϱ (where the series expansion should be used instead) five-point Lagrangian interpolation with respect to both variables will be sufficient and adequate.

6. Table I (Computed on the "BARK")

The irregular Coulomb wave function v, determined by:

$$\frac{d^2 v}{d \varrho^2} + \left(1 - \frac{2\alpha}{\varrho}\right) v = 0; \;\; v_{\varrho \to \infty} \sim \cos(\varrho + \eta);$$

$$\eta = -\alpha \log 2\varrho + \arg \Gamma(i\alpha + 1).$$

ϱ	v	v'	ϱ	v	v'
$\alpha = 0.01$					
0	+1.015916	$-\infty$	0.27	+0.973465	−0.271222
0.01	+1.014984	−0.0779315	0.28	+0.970707	−0.280236
0.02	+1.014232	−0.0740115	0.29	+0.967860	−0.289249
0.03	+1.013485	−0.0759282	0.30	+0.964922	−0.298258
0.04	+1.012706	−0.0802301	0.32	+0.958777	−0.316254
0.05	+1.011876	−0.0858352	0.34	+0.952273	−0.334206
0.055	+1.011440	−0.0889651	0.36	+0.945409	−0.352099
0.06	+1.010987	−0.0922614	0.38	+0.938189	−0.369917
0.065	+1.010517	−0.0956971	0.40	+0.930613	−0.387647
0.07	+1.010029	−0.0992511	0.42	+0.922684	−0.405276
0.075	+1.009524	−0.102907	0.44	+0.914403	−0.422793
0.08	+1.009000	−0.106650	0.46	+0.905773	−0.440186
0.085	+1.008458	−0.110471	0.48	+0.896797	−0.457445
0.09	+1.007895	−0.114359	0.50	+0.887476	−0.474560
0.095	+1.007314	−0.118308	0.52	+0.877815	−0.491521
0.10	+1.006712	−0.122310	0.54	+0.867817	−0.508319
0.105	+1.006091	−0.126360	0.56	+0.857484	−0.524945
0.11	+1.005449	−0.130453	0.58	+0.846820	−0.541391
0.115	+1.004786	−0.134585	0.60	+0.835829	−0.557647
0.12	+1.004103	−0.138752	0.65	+0.806947	−0.597409
0.125	+1.003398	−0.142951	0.70	+0.776111	−0.635820
0.13	+1.002673	−0.147180	0.75	+0.743390	−0.672767
0.135	+1.001927	−0.151435	0.80	+0.708860	−0.708143
0.14	+1.001159	−0.155714	0.85	+0.672603	−0.741848
0.145	+1.000370	−0.160015	0.90	+0.634705	−0.773790
0.15	+0.999559	−0.164337	0.95	+0.595255	−0.803880
0.16	+0.997872	−0.173035	1.00	+0.554349	−0.832036
0.17	+0.996098	−0.181796	1.05	+0.512085	−0.858182
0.18	+0.994236	−0.190610	1.10	+0.468566	−0.882246
0.19	+0.992285	−0.199469	1.15	+0.423896	−0.904166
0.20	+0.990246	−0.208365	1.20	+0.378186	−0.923880
0.21	+0.988118	−0.217291	1.25	+0.331546	−0.941337
0.22	+0.985900	−0.226243	1.30	+0.284091	−0.956490
0.23	+0.983593	−0.235215	1.40	+0.187201	−0.979723
0.24	+0.981196	−0.244203	1.50	+0.088467	−0.993327
0.25	+0.978709	−0.253203	1.60	−0.011139	−0.997146
0.26	+0.976132	−0.262210	1.70	−0.110634	−0.991125

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
1.8	−0.209037	−0.975311	4.0	−0.674436	+0.738379
1.9	−0.305375	−0.949847	4.1	−0.597368	+0.801708
2.0	−0.398693	−0.914976	4.2	−0.514360	+0.857072
2.1	−0.488067	−0.871034	4.3	−0.426237	+0.903920
2.2	−0.572611	−0.818449	4.4	−0.333873	+0.941782
2.3	−0.651486	−0.757738	4.5	−0.238193	+0.970280
2.4	−0.723908	−0.689497	4.6	−0.140141	+0.989129
2.5	−0.789157	−0.614400	4.7	−0.040695	+0.998140
2.6	−0.846584	−0.533187	4.8	+0.059155	+0.997220
2.7	−0.895617	−0.446662	4.9	+0.158418	+0.986377
2.8	−0.935768	−0.355684	5.0	+0.256104	+0.965717
2.9	−0.966636	−0.261153	5.1	+0.351241	+0.935445
3.0	−0.987911	−0.164008	5.2	+0.442882	+0.895860
3.1	−0.999382	−0.065213	5.3	+0.530115	+0.847355
3.2	−1.000932	+0.034250	5.4	+0.612072	+0.790412
3.3	−0.992543	+0.133393	5.5	+0.687935	+0.725597
3.4	−0.974297	+0.231228	5.6	+0.756950	+0.653553
3.5	−0.946373	+0.326784	5.7	+0.818429	+0.574997
3.6	−0.909048	+0.419109	5.8	+0.871759	+0.490712
3.7	−0.862690	+0.507284	5.9	+0.916408	+0.401535
3.8	−0.807759	+0.590430	6.0	+0.951932	+0.308355
3.9	−0.744799	+0.667718			
		$\alpha = 0.02$			
0	+1.032254	$-\infty$	0.22	+0.996086	−0.234884
0.01	+1.030412	−0.148076	0.23	+0.993697	−0.243064
0.02	+1.029039	−0.129822	0.24	+0.991224	−0.251300
0.03	+1.027779	−0.123426	0.25	+0.988670	−0.259583
0.04	+1.026555	−0.121877	0.26	+0.986033	−0.267907
0.05	+1.025332	−0.122979	0.27	+0.983312	−0.276267
0.055	+1.024715	−0.124196	0.28	+0.980507	−0.284658
0.06	+1.024090	−0.125753	0.29	+0.977619	−0.293075
0.065	+1.023457	−0.127594	0.30	+0.974646	−0.301512
0.07	+1.022813	−0.129677	0.32	+0.968446	−0.318436
0.075	+1.022160	−0.131967	0.34	+0.961908	−0.335399
0.08	+1.021494	−0.134439	0.36	+0.955030	−0.352377
0.085	+1.020815	−0.137068	0.38	+0.947813	−0.369349
0.09	+1.020123	−0.139837	0.40	+0.940257	−0.386293
0.095	+1.019416	−0.142731	0.42	+0.932362	−0.403192
0.10	+1.018695	−0.145735	0.44	+0.924129	−0.420030
0.105	+1.017959	−0.148839	0.46	+0.915561	−0.436792
0.11	+1.017206	−0.152034	0.48	+0.906658	−0.453464
0.115	+1.016438	−0.155310	0.50	+0.897423	−0.470032
0.12	+1.015653	−0.158661	0.52	+0.887858	−0.486485
0.125	+1.014852	−0.162079	0.54	+0.877965	−0.502811
0.13	+1.014032	−0.165560	0.56	+0.867746	−0.518999
0.135	+1.013196	−0.169098	0.58	+0.857206	−0.535038
0.14	+1.012341	−0.172688	0.60	+0.846346	−0.550919
0.15	+1.010578	−0.180011	0.62	+0.835170	−0.566632
0.16	+1.008740	−0.187502	0.64	+0.823682	−0.582168
0.17	+1.006828	−0.195136	0.66	+0.811885	−0.597517
0.18	+1.004837	−0.202894	0.68	+0.799782	−0.612672
0.19	+1.002769	−0.210761	0.70	+0.787379	−0.627624
0.20	+1.000622	−0.218723	0.72	+0.774679	−0.642365
0.21	+0.998394	−0.226768	0.74	+0.761686	−0.656887

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.76	+0.748405	−0.671183	1.8	−0.190751	−0.976414
0.78	+0.734840	−0.685245	1.9	−0.287301	−0.953009
0.80	+0.720997	−0.699067	2.0	−0.381040	−0.920250
0.85	+0.685200	−0.732523	2.1	−0.471049	−0.878442
0.90	+0.647772	−0.764329	2.2	−0.556440	−0.827981
0.95	+0.608797	−0.794390	2.3	−0.636373	−0.769353
1.00	+0.568363	−0.822617	2.4	−0.710057	−0.703122
1.05	+0.526567	−0.848927	2.5	−0.776765	−0.629935
1.10	+0.483504	−0.873243	2.6	−0.835836	−0.550504
1.15	+0.439277	−0.895497	2.7	−0.886684	−0.465608
1.20	+0.393990	−0.915623	2.8	−0.928803	−0.376081
1.25	+0.347751	−0.933564	2.9	−0.961775	−0.282802
1.3	+0.300671	−0.949269	3.0	−0.985269	−0.186691
1.4	+0.204440	−0.973794	3.2	−1.002978	+0.010216
1.5	+0.106226	−0.988909	3.6	−0.920762	+0.395519
1.6	+0.006978	−0.994426	4.0	−0.694782	+0.719501
1.7	−0.092337	−0.990257			
		$\alpha = 0.03$			
0	+1.049028	−∞	0.525	+0.895532	−0.485750
0.01	+1.046247	−0.220508	0.550	+0.883141	−0.505502
0.02	+1.044232	−0.187486	0.575	+0.870258	−0.525082
0.03	+1.042440	−0.172536	0.600	+0.856888	−0.544467
0.04	+1.040757	−0.164973	0.65	+0.828708	−0.582566
0.05	+1.039127	−0.161448	0.70	+0.798649	−0.619638
0.06	+1.037519	−0.160473	0.75	+0.766764	−0.655539
0.07	+1.035912	−0.161251	0.80	+0.733117	−0.690138
0.08	+1.034290	−0.163309	0.85	+0.697774	−0.723314
0.09	+1.032643	−0.166340	0.90	+0.660811	−0.754954
0.10	+1.030961	−0.170135	0.95	+0.622306	−0.784956
0.11	+1.029238	−0.174545	1.00	+0.582344	−0.813224
0.12	+1.027468	−0.179460	1.05	+0.541014	−0.839668
0.13	+1.025647	−0.184796	1.10	+0.498409	−0.864208
0.14	+1.023771	−0.190486	1.15	+0.454626	−0.886767
0.15	+1.021837	−0.196480	1.20	+0.409766	−0.907276
0.16	+1.019841	−0.202736	1.25	+0.363934	−0.925675
0.17	+1.017781	−0.209218	1.30	+0.317235	−0.941905
0.18	+1.015656	−0.215898	1.35	+0.269780	−0.955918
0.19	+1.013463	−0.222753	1.40	+0.221681	−0.967671
0.20	+1.011200	−0.229760	1.45	+0.173052	−0.977125
0.21	+1.008867	−0.236904	1.50	+0.124007	−0.984251
0.22	+1.006462	−0.244168	1.55	+0.074666	−0.989023
0.23	+1.003983	−0.251539	1.60	+0.025145	−0.991423
0.24	+1.001431	−0.259006	1.65	−0.024437	−0.991440
0.25	+0.998803	−0.266557	1.70	−0.073960	−0.989067
0.26	+0.996099	−0.274185	1.75	−0.123304	−0.984306
0.27	+0.993319	−0.281879	1.80	−0.172351	−0.977163
0.28	+0.990461	−0.289634	1.85	−0.220981	−0.967651
0.29	+0.987526	−0.297442	1.90	−0.269076	−0.955789
0.30	+0.984512	−0.305296	1.95	−0.316521	−0.941602
0.325	+0.976633	−0.325102	2.00	−0.363198	−0.925122
0.350	+0.968256	−0.345090	2.4	−0.695851	−0.716335
0.375	+0.959377	−0.365196	2.8	−0.921333	−0.396170
0.400	+0.949996	−0.385367	3.0	−0.982069	−0.209157
0.425	+0.940109	−0.405556	3.2	−1.004434	−0.013713
0.450	+0.929718	−0.425724	3.6	−0.931897	+0.371769
0.475	+0.918823	−0.445833	4.0	−0.714667	+0.700186
0.500	+0.907427	−0.465851			

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
		$\alpha = 0.04$			
0	+1.066249	−∞	0.575	+0.880679	−0.519424
0.01	+1.062498	−0.295303	0.600	+0.867457	−0.538300
0.02	+1.059820	−0.247065	0.625	+0.853765	−0.557006
0.03	+1.057480	−0.223309	0.650	+0.839608	−0.575517
0.04	+1.055321	−0.209560	0.675	+0.824991	−0.593812
0.05	+1.053271	−0.201281	0.700	+0.809919	−0.611871
0.06	+1.051284	−0.196455	0.725	+0.794400	−0.629674
0.07	+1.049334	−0.194006	0.750	+0.778438	−0.647202
0.08	+1.047398	−0.193290	0.775	+0.762042	−0.664438
0.09	+1.045463	−0.193894	0.800	+0.745219	−0.681365
0.10	+1.043517	−0.195535	0.825	+0.727976	−0.697968
0.11	+1.041550	−0.198011	0.850	+0.710323	−0.714230
0.12	+1.039554	−0.201173	0.875	+0.692268	−0.730136
0.13	+1.037524	−0.204908	0.900	+0.673819	−0.745674
0.14	+1.035455	−0.209128	0.925	+0.654987	−0.760828
0.15	+1.033340	−0.213763	0.950	+0.635781	−0.775587
0.16	+1.031178	−0.218756	0.975	+0.616211	−0.789936
0.17	+1.028964	−0.224061	1.000	+0.596288	−0.803865
0.18	+1.026696	−0.229639	1.025	+0.576022	−0.817362
0.19	+1.024371	−0.235458	1.050	+0.555424	−0.830415
0.20	+1.021986	−0.241492	1.075	+0.534505	−0.843013
0.21	+1.019540	−0.247715	1.100	+0.513277	−0.855148
0.22	+1.017031	−0.254108	1.125	+0.491751	−0.866807
0.23	+1.014457	−0.260653	1.150	+0.469941	−0.877984
0.24	+1.011818	−0.267335	1.175	+0.447856	−0.888667
0.25	+1.009110	−0.274140	1.200	+0.425511	−0.898849
0.26	+1.006334	−0.281055	1.225	+0.402918	−0.908522
0.27	+1.003489	−0.288070	1.250	+0.380090	−0.917677
0.28	+1.000573	−0.295176	1.275	+0.357039	−0.926307
0.29	+0.997585	−0.302362	1.300	+0.333779	−0.934406
0.30	+0.994525	−0.309621	1.325	+0.310323	−0.941967
0.325	+0.986555	−0.328042	1.350	+0.286685	−0.948984
0.350	+0.978121	−0.346777	1.375	+0.262878	−0.955450
0.375	+0.969214	−0.365745	1.400	+0.238917	−0.961361
0.400	+0.959832	−0.384879	1.425	+0.214815	−0.966712
0.425	+0.949970	−0.404120	1.450	+0.190586	−0.971497
0.450	+0.939625	−0.423421	1.475	+0.166245	−0.975714
0.475	+0.928798	−0.442736	1.500	+0.141805	−0.979358
0.500	+0.917489	−0.462027	1.750	−0.104803	−0.983738
0.525	+0.905698	−0.481259	2.000	−0.345176	−0.929592
0.550	+0.893427	−0.500401			
		$\alpha = 0.06$			
0	+1.102086	−∞	0.10	+1.069533	−0.249438
0.01	+1.096300	−0.452297	0.11	+1.067047	−0.247903
0.02	+1.092230	−0.372207	0.12	+1.064572	−0.247432
0.03	+1.088736	−0.330049	0.13	+1.062096	−0.247852
0.04	+1.085578	−0.303388	0.14	+1.059612	−0.249026
0.05	+1.082641	−0.285198	0.15	+1.057113	−0.250847
0.06	+1.079857	−0.272354	0.16	+1.054593	−0.253229
0.07	+1.077181	−0.263188	0.17	+1.052047	−0.256099
0.08	+1.074584	−0.256706	0.18	+1.049470	−0.259399
0.09	+1.072041	−0.252269	0.19	+1.046858	−0.263080

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.20	+1.044207	−0.267100	0.550	+0.914134	−0.491195
0.225	+1.037392	−0.278409	0.575	+0.901630	−0.509050
0.250	+1.030274	−0.291183	0.600	+0.888682	−0.526857
0.275	+1.022823	−0.305106	0.625	+0.875288	−0.544587
0.300	+1.015011	−0.319940	0.650	+0.861453	−0.562210
0.325	+1.006820	−0.335502	0.675	+0.847179	−0.579699
0.350	+0.998231	−0.351650	0.700	+0.832469	−0.597030
0.375	+0.989233	−0.368266	0.750	+0.801762	−0.631126
0.400	+0.979815	−0.385255	0.800	+0.769372	−0.664325
0.425	+0.969968	−0.402535	0.850	+0.735347	−0.696474
0.450	+0.959686	−0.420039	0.900	+0.699744	−0.727434
0.475	+0.948965	−0.437705	0.950	+0.662626	−0.757078
0.500	+0.937800	−0.455487	1.000	+0.624060	−0.785289
0.525	+0.926190	−0.473327			
$\alpha = 0.08$					
0	+1.139873	$-\infty$	0.300	+1.036143	−0.332565
0.01	+1.131915	−0.619720	0.325	+1.027674	−0.345147
0.02	+1.126360	−0.505760	0.350	+1.018879	−0.358595
0.03	+1.121636	−0.444074	0.375	+1.009738	−0.372756
0.04	+1.117409	−0.403735	0.400	+1.000236	−0.387503
0.05	+1.113523	−0.375062	0.425	+0.990359	−0.402731
0.06	+1.109884	−0.353747	0.450	+0.980096	−0.418352
0.07	+1.106431	−0.337496	0.475	+0.969439	−0.434289
0.08	+1.103122	−0.324940	0.500	+0.958380	−0.450476
0.09	+1.099923	−0.315196	0.525	+0.946913	−0.466855
0.10	+1.096810	−0.307663	0.550	+0.935036	−0.483376
0.11	+1.093764	−0.301913	0.575	+0.922744	−0.499992
0.12	+1.090767	−0.297629	0.600	+0.910036	−0.516662
0.13	+1.087807	−0.294572	0.625	+0.896910	−0.533348
0.14	+1.084872	−0.292554	0.650	+0.883368	−0.550016
0.15	+1.081953	−0.291428	0.675	+0.869410	−0.566634
0.16	+1.079041	−0.291076	0.700	+0.855037	−0.583173
0.17	+1.076129	−0.291399	0.750	+0.825058	−0.615906
0.18	+1.073211	−0.292317	0.800	+0.793457	−0.648016
0.19	+1.070281	−0.293763	0.850	+0.760270	−0.679327
0.20	+1.067334	−0.295679	0.900	+0.725540	−0.709682
0.225	+1.059865	−0.302224	0.950	+0.689320	−0.738937
0.250	+1.052206	−0.310822	1.000	+0.651667	−0.766963
0.275	+1.044311	−0.321042			
$\alpha = 0.10$					
0	+1.179720	$-\infty$	0.10	+1.125418	−0.370439
0.01	+1.169448	−0.798275	0.11	+1.121766	−0.360257
0.02	+1.162309	−0.648268	0.12	+1.118207	−0.351966
0.03	+1.156270	−0.565839	0.13	+1.114722	−0.345257
0.04	+1.150902	−0.510996	0.14	+1.111297	−0.339890
0.05	+1.146001	−0.471224	0.15	+1.107920	−0.335675
0.055	+1.143686	−0.455125	0.16	+1.104580	−0.332458
0.06	+1.141446	−0.440954	0.17	+1.101268	−0.330114
0.065	+1.139274	−0.428400	0.18	+1.097975	−0.328540
0.07	+1.137160	−0.417221	0.19	+1.094695	−0.327648
0.08	+1.133086	−0.398256	0.20	+1.091420	−0.327365
0.09	+1.129183	−0.382921	0.225	+1.083223	−0.328933

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.250	+1.074952	−0.333170	0.575	+0.944037	−0.492322
0.275	+1.066547	−0.339527	0.600	+0.931536	−0.507784
0.300	+1.057961	−0.347596	0.625	+0.918647	−0.523359
0.325	+1.049155	−0.357069	0.650	+0.905367	−0.539005
0.350	+1.040098	−0.367701	0.675	+0.891696	−0.554687
0.375	+1.030762	−0.379298	0.700	+0.877633	−0.570369
0.400	+1.021126	−0.391704	0.750	+0.848332	−0.601612
0.425	+1.011172	−0.404787	0.800	+0.817477	−0.632510
0.450	+1.000882	−0.418437	0.850	+0.785091	−0.662861
0.475	+0.990246	−0.432560	0.900	+0.751203	−0.692490
0.500	+0.979251	−0.447077	0.950	+0.715856	−0.721238
0.525	+0.967889	−0.461916	1.000	+0.679097	−0.748959
0.550	+0.956153	−0.477016			
$\alpha = 0.12$					
0	+1.221749	−∞	0.275	+1.089578	−0.360668
0.01	+1.209007	−0.988717	0.300	+1.080510	−0.365134
0.02	+1.200178	−0.800312	0.325	+1.071307	−0.371362
0.03	+1.192737	−0.695831	0.350	+1.061929	−0.379056
0.04	+1.186150	−0.625594	0.375	+1.052344	−0.387979
0.05	+1.180164	−0.574058	0.400	+1.042522	−0.397940
0.055	+1.177347	−0.552988	0.425	+1.032439	−0.408782
0.06	+1.174630	−0.534310	0.450	+1.022076	−0.420371
0.065	+1.172001	−0.517637	0.475	+1.011416	−0.432596
0.07	+1.169451	−0.502668	0.500	+1.000442	−0.445360
0.08	+1.164557	−0.476937	0.525	+0.989144	−0.458581
0.09	+1.159897	−0.455704	0.550	+0.977510	−0.472186
0.10	+1.155431	−0.438007	0.575	+0.965532	−0.486109
0.11	+1.151127	−0.423158	0.600	+0.953202	−0.500294
0.12	+1.146960	−0.410653	0.625	+0.940515	−0.514689
0.13	+1.142908	−0.400108	0.650	+0.927466	−0.529247
0.14	+1.138952	−0.391224	0.675	+0.914052	−0.543926
0.15	+1.135079	−0.383767	0.700	+0.900269	−0.558687
0.16	+1.131273	−0.377546	0.750	+0.871595	−0.588316
0.17	+1.127524	−0.372407	0.800	+0.841439	−0.617877
0.18	+1.123822	−0.368222	0.850	+0.809812	−0.647147
0.19	+1.120157	−0.364882	0.900	+0.776732	−0.675931
0.20	+1.116522	−0.362298	0.950	+0.742230	−0.704052
0.225	+1.107517	−0.358662	1.000	+0.706341	−0.731353
0.250	+1.098561	−0.358344			
$\alpha = 0.16$					
0	+1.312860	−∞	0.10	+1.219986	−0.588554
0.01	+1.294671	−1.408530	0.11	+1.214228	−0.563603
0.02	+1.282117	−1.135526	0.12	+1.208703	−0.541985
0.03	+1.271583	−0.982594	0.13	+1.203379	−0.523154
0.04	+1.262306	−0.878617	0.14	+1.198232	−0.506685
0.05	+1.253923	−0.801354	0.15	+1.193239	−0.492242
0.055	+1.249997	−0.769430	0.16	+1.188381	−0.479557
0.06	+1.246223	−0.740918	0.17	+1.183643	−0.468408
0.065	+1.242584	−0.715266	0.18	+1.179009	−0.458614
0.07	+1.239066	−0.692044	0.19	+1.174466	−0.450022
0.08	+1.232354	−0.651597	0.20	+1.170005	−0.442503
0.09	+1.226013	−0.617559	0.225	+1.159138	−0.427720

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.250	+1.148579	−0.417661	0.575	+1.009226	−0.478336
0.275	+1.138224	−0.411371	0.600	+0.997126	−0.489753
0.300	+1.127985	−0.408146	0.625	+0.984735	−0.501581
0.325	+1.117795	−0.407456	0.650	+0.972044	−0.513761
0.350	+1.107595	−0.408884	0.675	+0.959044	−0.526239
0.375	+1.097336	−0.412105	0.700	+0.945730	−0.538965
0.400	+1.086977	−0.416852	0.750	+0.918134	−0.564986
0.425	+1.076483	−0.422909	0.800	+0.889223	−0.591507
0.450	+1.065822	−0.430095	0.850	+0.858979	−0.618255
0.475	+1.054970	−0.438258	0.900	+0.827397	−0.644993
0.500	+1.043902	−0.447267	0.950	+0.794483	−0.671510
0.525	+1.032600	−0.457013	1.000	+0.760253	−0.697620
0.550	+1.021046	−0.467398			

$\alpha = 0.20$

ϱ	v	v'	ϱ	v	v'
0	+1.414301	$-\infty$	0.275	+1.190669	−0.474091
0.01	+1.389905	−1.886273	0.300	+1.178970	−0.462472
0.02	+1.373113	−1.516866	0.325	+1.167517	−0.454237
0.03	+1.359061	−1.308916	0.350	+1.156234	−0.448840
0.04	+1.346721	−1.166742	0.375	+1.145055	−0.445850
0.05	+1.335608	−1.060434	0.400	+1.133925	−0.444919
0.055	+1.330418	−1.016278	0.425	+1.122795	−0.445764
0.06	+1.325437	−0.976699	0.450	+1.111624	−0.448150
0.065	+1.320645	−0.940953	0.475	+1.100376	−0.451880
0.07	+1.316022	−0.908465	0.500	+1.089020	−0.456786
0.08	+1.307231	−0.851521	0.525	+1.077528	−0.462726
0.09	+1.298963	−0.803156	0.550	+1.065876	−0.469575
0.10	+1.291145	−0.761526	0.575	+1.054043	−0.477227
0.11	+1.283715	−0.725317	0.600	+1.042009	−0.485585
0.12	+1.276624	−0.693562	0.625	+1.029758	−0.494567
0.13	+1.269831	−0.665528	0.650	+1.017276	−0.504098
0.14	+1.263303	−0.640648	0.675	+1.004549	−0.514109
0.15	+1.257009	−0.618472	0.700	+0.991567	−0.524541
0.16	+1.250925	−0.598639	0.750	+0.964797	−0.546453
0.17	+1.245029	−0.580855	0.800	+0.936903	−0.569448
0.18	+1.239302	−0.564876	0.850	+0.907840	−0.593198
0.19	+1.233727	−0.550499	0.900	+0.877576	−0.617422
0.20	+1.228287	−0.537552	0.950	+0.846094	−0.641875
0.225	+1.215201	−0.510530	1.000	+0.813388	−0.666338
0.250	+1.202708	−0.489800			

$\alpha = 0.25$

ϱ	v	v'	ϱ	v	v'
0	+1.557506	$-\infty$	0.085	+1.406383	−1.105772
0.010	+1.524114	−2.577392	0.090	+1.400938	1.072675
0.020	+1.501195	−2.067957	0.095	+1.395653	−1.041865
0.030	+1.482058	−1.780387	0.100	+1.390516	−1.013102
0.040	+1.465293	−1.583113	0.105	+1.385519	−0.986181
0.050	+1.450234	−1.435026	0.110	+1.380652	−0.960925
0.055	+1.443216	−1.373314	0.115	+1.375907	−0.937183
0.060	+1.436490	1.317871	0.120	+1.371277	−0.914821
0.065	+1.430028	−1.267675	0.125	+1.366757	−0.893723
0.070	+1.423806	−1.221936	0.130	+1.362338	−0.873786
0.075	+1.417803	−1.180026	0.135	+1.358017	−0.854920
0.080	+1.412000	−1.141443	0.140	+1.353787	−0.837044

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.150	+1.345585	−0.803981	0.425	+1.184408	−0.487632
0.160	+1.337697	−0.774102	0.450	+1.172274	−0.483412
0.170	+1.330094	−0.747007	0.475	+1.160222	−0.481039
0.180	+1.322749	−0.722363	0.500	+1.148209	−0.480292
0.190	+1.315639	−0.699892	0.550	+1.124149	−0.482947
0.200	+1.308745	−0.679360	0.600	+1.099838	−0.490161
0.225	+1.292332	−0.635276	0.650	+1.075072	−0.501006
0.250	+1.276910	−0.599712	0.700	+1.049689	−0.514755
0.275	+1.262289	−0.570945	0.750	+1.023558	−0.530822
0.300	+1.248317	−0.547712	0.800	+0.996576	−0.548728
0.325	+1.234866	−0.529058	0.850	+0.968662	−0.568071
0.350	+1.221832	−0.514249	0.900	+0.939751	−0.588508
0.375	+1.209126	−0.502704	0.950	+0.909798	−0.609745
0.400	+1.196673	−0.493958	1.000	+0.878768	−0.631523
$\alpha = 0.3$					
0	+1.721481	$-\infty$	0.24	+1.366822	−0.753576
0.01	+1.677482	−3.390212	0.25	+1.359387	−0.733820
0.02	+1.647363	−2.666471	0.26	+1.352141	−0.715473
0.03	+1.622260	−2.333460	0.27	+1.345073	−0.698420
0.04	+1.600305	−2.071395	0.28	+1.338169	−0.682561
0.05	+1.580618	−1.874330	0.29	+1.331418	−0.667805
0.055	+1.571456	−1.792080	0.30	+1.324809	−0.654071
0.06	+1.562683	−1.718102	0.325	+1.308848	−0.623746
0.065	+1.554263	−1.651046	0.350	+1.293580	−0.598415
0.07	+1.546163	−1.589866	0.375	+1.278892	−0.577324
0.075	+1.538356	−1.533734	0.400	+1.264684	−0.559869
0.08	+1.530818	−1.481982	0.425	+1.250872	−0.545559
0.085	+1.523530	−1.434066	0.450	+1.237383	−0.533993
0.09	+1.516472	−1.389537	0.475	+1.224153	−0.524834
0.095	+1.509629	−1.348018	0.500	+1.211124	−0.517799
0.10	+1.502987	−1.309191	0.525	+1.198247	−0.512649
0.11	+1.490256	−1.238568	0.550	+1.185477	−0.509177
0.12	+1.478189	−1.175921	0.575	+1.172776	−0.507205
0.13	+1.466715	−1.119929	0.600	+1.160106	−0.506579
0.14	+1.455772	−1.069566	0.625	+1.147436	−0.507163
0.15	+1.445308	−1.024024	0.650	+1.134739	−0.508837
0.16	+1.435278	−0.982653	0.675	+1.121986	−0.511494
0.17	+1.425643	−0.944924	0.700	+1.109156	−0.515040
0.18	+1.416369	−0.910400	0.750	+1.083181	−0.524466
0.19	+1.407425	−0.878715	0.800	+1.056666	−0.536526
0.20	+1.398786	−0.849564	0.850	+1.029492	−0.550734
0.21	+1.390426	−0.822684	0.900	+1.001564	−0.566680
0.22	+1.382325	−0.797850	0.950	+0.972802	−0.584011
0.23	+1.374463	−0.774870	1.000	+0.943145	−0.602425
$\alpha = 0.4$					
0	+2.124652	$-\infty$	0.06	+1.869347	−2.734350
0.01	+2.053360	−5.471696	0.065	+1.855954	−2.624387
0.02	+2.004840	−4.365763	0.07	+1.843087	−2.523980
0.03	+1.964531	−3.741555	0.075	+1.830701	−2.431777
0.04	+1.929373	−3.312827	0.08	+1.818757	−2.346686
0.05	+1.897925	−2.990289	0.085	+1.807224	−2.267820
0.055	+1.883316	−2.855581	0.09	+1.796070	−2.194444

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.095	+1.785271	−2.125943	0.325	+1.480967	−0.881059
0.10	+1.774803	−2.061799	0.350	+1.459586	−0.830645
0.11	+1.754782	−1.944879	0.375	+1.439380	−0.786875
0.12	+1.735863	−1.840837	0.400	+1.420195	−0.748795
0.13	+1.717929	−1.747523	0.425	+1.401899	−0.715633
0.14	+1.700882	−1.663269	0.450	+1.384378	−0.686755
0.15	+1.684638	−1.586764	0.475	+1.367530	−0.661637
0.16	+1.669125	−1.516952	0.500	+1.351268	−0.639837
0.17	+1.654280	−1.452976	0.525	+1.335514	−0.620984
0.18	+1.640048	−1.394126	0.550	+1.320197	−0.604761
0.19	+1.626382	−1.339815	0.575	+1.305256	−0.590896
0.20	+1.613238	−1.289543	0.600	+1.290635	−0.579151
0.21	+1.600579	−1.242890	0.625	+1.276283	−0.569321
0.22	+1.588370	−1.199494	0.650	+1.262154	−0.561226
0.23	+1.576579	−1.159044	0.675	+1.248208	−0.554708
0.24	+1.565180	−1.121267	0.700	+1.234407	−0.549625
0.25	+1.554146	−1.085928	0.750	+1.207104	−0.543275
0.26	+1.543454	−1.052820	0.800	+1.180006	−0.541321
0.27	+1.533083	−1.021758	0.850	+1.152911	−0.543065
0.28	+1.523013	−0.992581	0.900	+1.125648	−0.547927
0.29	+1.513225	−0.965143	0.950	+1.098074	−0.555423
0.30	+1.503704	−0.939316	1.000	+1.070068	−0.565137
$\alpha = 0.5$					
0	+2.654732	$-\infty$	0.102	+2.118242	−3.047130
0.01	+2.545373	−8.245180	0.104	+2.112185	−3.010287
0.02	+2.471438	−6.639616	0.106	+2.106200	−2.974329
0.03	+2.410225	−5.673831	0.108	+2.100287	−2.939221
0.04	+2.356974	−5.011734	0.110	+2.094443	−2.904931
0.05	+2.309448	−4.514316	0.112	+2.088666	−2.871427
0.052	+2.300506	−4.428523	0.114	+2.082957	−2.838681
0.054	+2.291732	−4.346458	0.116	+2.077311	−2.806664
0.056	+2.283118	−4.267844	0.118	+2.071729	−2.775350
0.058	+2.274658	−4.192432	0.120	+2.066209	−2.744715
0.060	+2.266346	−4.119999	0.13	+2.039494	−2.600922
0.062	+2.258177	−4.050344	0.14	+2.014145	−2.470982
0.064	+2.250143	−3.983286	0.15	+1.990035	−2.352869
0.066	+2.242242	−3.918659	0.16	+1.967053	−2.244959
0.068	+2.234467	−3.856314	0.17	+1.945106	−2.145931
0.070	+2.226815	−3.796114	0.18	+1.924109	−2.054695
0.072	+2.219281	−3.737935	0.19	+1.903989	−1.970347
0.074	+2.211862	−3.681661	0.20	+1.884682	−1.892122
0.076	+2.204553	−3.627188	0.21	+1.866128	−1.819373
0.078	+2.197352	−3.574419	0.22	+1.848278	−1.751547
0.080	+2.190254	−3.523264	0.23	+1.831083	−1.688166
0.082	+2.183258	−3.473641	0.24	+1.814501	−1.628815
0.084	+2.176359	−3.425472	0.25	+1.798494	−1.573135
0.086	+2.169555	−3.378687	0.26	+1.783027	−1.520807
0.088	+2.162843	−3.333220	0.27	+1.768068	−1.471552
0.090	+2.156221	−3.289008	0.28	+1.753587	−1.425122
0.092	+2.149686	−3.245994	0.29	+1.739557	−1.381298
0.094	+2.143236	−3.204125	0.30	+1.725953	−1.339882
0.096	+2.136869	−3.163349	0.325	+1.693659	−1.245763
0.098	+2.130582	−3.123621	0.350	+1.663567	−1.163325
0.100	+2.124374	−3.084895	0.375	+1.635410	−1.090754

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.400	+1.608959	− 1.026612	0.650	+1.405332	−0.669403
0.425	+1.584019	− 0.969734	0.675	+1.388824	−0.651602
0.450	+1.560420	− 0.919171	0.700	+1.372734	−0.635904
0.475	+1.538015	− 0.874137	0.750	+1.341613	−0.610118
0.500	+1.516673	− 0.833975	0.800	+1.311614	−0.590821
0.525	+1.496280	− 0.798134	0.850	+1.282439	−0.577032
0.550	+1.476734	− 0.766142	0.900	+1.253833	−0.567947
0.575	+1.457944	− 0.737598	0.950	+1.225577	−0.562898
0.600	+1.439828	− 0.712154	1.000	+1.197485	−0.561325
0.625	+1.422313	− 0.689510			
$\alpha = 0.6$					
0	+3.352706	−∞	0.126	+2.467076	−3.833850
0.01	+3.190209	−12.352005	0.128	+2.459450	−3.792226
0.02	+3.081161	− 9.771698	0.130	+2.451906	−3.751449
0.03	+2.991217	− 8.323873	0.132	+2.444443	−3.711492
0.04	+2.913194	− 7.333870	0.134	+2.437060	−3.672329
0.05	+2.843724	− 6.591696	0.136	+2.429754	−3.633934
0.052	+2.830670	− 6.463837	0.138	+2.422524	−3.596284
0.054	+2.817865	− 6.341578	0.140	+2.415368	−3.559355
0.056	+2.805300	− 6.224500	0.142	+2.408286	−3.523126
0.058	+2.792964	− 6.112227	0.144	+2.401275	−3.487575
0.060	+2.780848	− 6.004424	0.146	+2.394335	−3.452682
0.062	+2.768944	− 5.900787	0.148	+2.387464	−3.418428
0.064	+2.757242	− 5.801043	0.150	+2.380661	−3.384794
0.066	+2.745737	− 5.704944	0.16	+2.347622	−3.225337
0.068	+2.734420	− 5.612265	0.17	+2.316111	−3.079010
0.070	+2.723286	− 5.522799	0.18	+2.286004	−2.944187
0.072	+2.712327	− 5.436358	0.19	+2.257193	−2.819518
0.074	+2.701538	− 5.352771	0.20	+2.229583	−2.703865
0.076	+2.690914	− 5.271879	0.21	+2.203089	−2.596263
0.078	+2.680449	− 5.193535	0.22	+2.177634	−2.495891
0.080	+2.670139	− 5.117607	0.23	+2.153150	−2.402036
0.082	+2.659977	− 5.043968	0.24	+2.129574	−2.314086
0.084	+2.649961	− 4.972504	0.25	+2.106850	−2.231504
0.086	+2.640086	− 4.903107	0.26	+2.084927	−2.153819
0.088	+2.630348	− 4.835679	0.27	+2.063758	−2.080618
0.090	+2.620742	− 4.770125	0.28	+2.043301	−2.011534
0.092	+2.611266	− 4.706361	0.29	+2.023515	−1.946241
0.094	+2.601916	− 4.644304	0.300	+2.004364	−1.884449
0.096	+2.592688	− 4.583880	0.325	+1.959052	−1.743638
0.098	+2.583579	− 4.525018	0.350	+1.917042	−1.619733
0.100	+2.574587	− 4.467650	0.375	+1.877946	−1.510069
0.102	+2.565707	− 4.411716	0.400	+1.841437	−1.412530
0.104	+2.556939	− 4.357155	0.425	+1.807233	−1.325415
0.106	+2.548278	− 4.303913	0.450	+1.775091	−1.247335
0.108	+2.539722	− 4.251938	0.475	+1.744801	−1.177144
0.110	+2.531270	− 4.201180	0.500	+1.716176	−1.113889
0.112	+2.522917	− 4.151592	0.525	+1.689055	−1.056767
0.114	+2.514663	− 4.103132	0.550	+1.663293	−1.005098
0.116	+2.506504	− 4.055757	0.575	+1.638760	−0.958302
0.118	+2.498439	− 4.009428	0.600	+1.615341	−0.915881
0.120	+2.490465	− 3.964107	0.625	+1.592933	−0.877401
0.122	+2.482582	− 3.919760	0.650	+1.571441	−0.842489
0.124	+2.474786	− 3.876351	0.675	+1.550781	−0.810815

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.700	+1.530876	−0.782091	0.900	+1.391443	−0.633041
0.750	+1.493052	−0.732500	0.950	+1.360325	−0.612560
0.800	+1.457475	−0.691999	1.000	+1.330105	−0.597013
0.850	+1.423724	−0.659217			
		$\alpha = 0.7$			
0	+4.273043	$-\infty$	0.134	+2.960535	−5.181672
0.01	+4.036358	−17.892774	0.136	+2.950227	−5.126285
0.02	+3.878794	−14.086104	0.138	+2.940029	−5.071982
0.03	+3.749365	−11.957683	0.140	+2.929939	−5.018729
0.04	+3.637437	−10.506476	0.142	+2.919954	−4.966494
0.05	+3.538032	−9.421237	0.144	+2.910072	−4.915246
0.052	+3.519377	−9.234535	0.146	+2.900292	−4.864955
0.054	+3.501088	−9.056086	0.148	+2.890612	−4.815593
0.056	+3.483148	−8.885268	0.150	+2.881029	−4.767134
0.058	+3.465542	−8.721529	0.16	+2.834523	−4.537493
0.060	+3.448257	−8.564370	0.17	+2.790216	−4.326912
0.062	+3.431280	−8.413343	0.18	+2.747929	−4.133012
0.064	+3.414600	−8.268044	0.19	+2.707506	−3.953813
0.066	+3.398205	−8.128107	0.20	+2.668809	−3.787652
0.068	+3.382084	−7.993198	0.21	+2.631715	−3.633123
0.070	+3.366229	−7.863014	0.22	+2.596112	−3.489021
0.072	+3.350629	−7.737277	0.23	+2.561903	−3.354311
0.074	+3.335277	−7.615731	0.24	+2.528997	−3.228096
0.076	+3.320163	−7.498143	0.25	+2.497315	−3.109598
0.078	+3.305282	−7.384299	0.26	+2.466782	−2.998130
0.080	+3.290624	−7.273999	0.27	+2.437331	−2.893091
0.082	+3.276183	−7.167059	0.28	+2.408901	−2.793949
0.084	+3.261954	−7.063310	0.29	+2.381434	−2.700231
0.086	+3.247928	−6.962593	0.30	+2.354879	−2.611513
0.088	+3.234101	−6.864761	0.325	+2.292175	−2.409217
0.090	+3.220467	−6.769679	0.350	+2.234218	−2.230980
0.092	+3.207021	−6.677217	0.375	+2.180458	−2.072950
0.094	+3.193757	−6.587258	0.400	+2.130428	−1.932079
0.096	+3.180670	−6.499690	0.425	+2.083732	−1.805914
0.098	+3.167756	−6.414408	0.450	+2.040027	−1.692462
0.100	+3.155011	−6.331315	0.475	+1.999017	−1.590081
0.102	+3.142430	−6.250318	0.500	+1.960442	−1.497406
0.104	+3.130008	−6.171331	0.525	+1.924075	−1.413292
0.106	+3.117743	−6.094273	0.550	+1.889714	−1.336769
0.108	+3.105630	−6.019066	0.575	+1.857180	−1.267014
0.110	+3.093666	−5.945639	0.600	+1.826313	−1.203318
0.112	+3.081846	−5.873923	0.625	+1.796969	−1.145071
0.114	+3.070169	−5.803853	0.650	+1.769018	−1.091742
0.116	+3.058630	−5.735368	0.675	+1.742344	−1.042868
0.118	+3.047226	−5.668411	0.700	+1.716841	−0.998045
0.120	+3.035955	−5.602925	0.750	+1.668967	−0.919161
0.122	+3.024814	−5.538860	0.800	+1.624718	−0.852694
0.124	+3.013799	−5.476165	0.850	+1.583523	−0.796742
0.126	+3.002908	−5.414793	0.900	+1.544895	−0.749772
0.128	+2.992139	−5.354700	0.950	+1.508417	−0.710538
0.130	+2.981489	−5.295842	1.000	+1.473729	−0.678009
0.132	+2.970955	−5.238179			

(Tabl I, Cont.)

ϱ	v	v'	ϱ	v	v'
		$\alpha = 0.8$			
0	+5.488285	−∞	0.134	+3.633081	−7.202691
0.01	+5.148074	−25.572431	0.136	+3.618755	−7.123994
0.02	+4.923479	−20.028614	0.138	+3.604585	−7.046856
0.03	+4.739790	−16.940050	0.140	+3.590567	−6.971228
0.04	+4.581461	−14.840451	0.142	+3.576699	−6.897062
0.050	+4.441227	−13.274404	0.144	+3.562978	−6.824316
0.052	+4.414949	−13.005381	0.146	+3.549401	−6.752946
0.054	+4.389197	−12.748365	0.148	+3.535965	−6.682910
0.056	+4.363949	−12.502451	0.150	+3.522668	−6.614170
0.058	+4.339181	−12.266829	0.160	+3.458179	−6.288637
0.060	+4.314875	−12.040772	0.170	+3.396804	−5.990439
0.062	+4.291012	−11.823629	0.180	+3.338290	−5.716134
0.064	+4.267575	−11.614807	0.190	+3.282412	−5.462861
0.066	+4.244547	−11.413772	0.200	+3.228971	−5.228221
0.068	+4.221915	−11.220038	0.210	+3.177792	−5.010181
0.070	+4.199663	−11.033162	0.220	+3.128718	−4.807007
0.072	+4.177778	−10.852737	0.230	+3.081607	−4.617208
0.074	+4.156248	−10.678395	0.240	+3.036334	−4.439494
0.076	+4.135060	−10.509795	0.250	+2.992781	−4.272741
0.078	+4.114205	−10.346620	0.260	+2.950845	−4.115967
0.080	+4.093670	−10.188583	0.270	+2.910431	−3.968307
0.082	+4.073447	−10.035415	0.280	+2.871451	−3.828996
0.084	+4.053526	−9.886869	0.290	+2.833826	−3.697357
0.086	+4.033890	−9.742716	0.300	+2.797481	−3.572784
0.088	+4.014552	−9.602738	0.325	+2.711788	−3.288857
0.090	+3.995483	−9.466743	0.350	+2.632756	−3.038795
0.092	+3.976683	−9.334538	0.375	+2.559612	−2.817103
0.094	+3.958143	−9.205955	0.400	+2.491701	−2.619436
0.096	+3.939857	−9.080829	0.425	+2.428469	−2.442307
0.098	+3.921817	−8.959007	0.450	+2.369438	−2.282885
0.100	+3.904018	−8.840350	0.475	+2.314197	−2.138842
0.102	+3.886454	−8.724723	0.500	+2.262385	−2.008248
0.104	+3.869118	−8.612000	0.525	+2.213686	−1.889486
0.106	+3.852004	−8.502063	0.550	+2.167823	−1.781191
0.108	+3.835107	−8.394798	0.575	+2.124549	−1.682201
0.110	+3.818423	−8.290104	0.600	+2.083643	−1.591522
0.112	+3.801945	−8.187877	0.625	+2.044910	−1.508299
0.114	+3.785670	−8.088026	0.650	+2.008173	−1.431789
0.116	+3.769592	−7.990461	0.675	+1.973270	−1.361347
0.118	+3.753706	−7.895100	0.700	+1.940059	−1.296407
0.120	+3.738010	−7.801860	0.750	+1.878198	−1.181107
0.122	+3.722498	−7.710666	0.800	+1.821670	−1.082567
0.124	+3.707166	−7.621448	0.850	+1.769706	−0.998162
0.126	+3.692011	−7.534135	0.900	+1.721653	−0.925789
0.128	+3.677028	−7.448664	0.950	+1.676955	−0.863744
0.130	+3.662215	−7.364972	1.000	+1.635130	−0.810628
0.132	+3.647567	−7.283000			
		$\alpha = 0.9$			
0	+7.095223	−∞	0.05	+5.617500	−18.521178
0.01	+6.610649	−36.211607	0.052	+5.580844	−18.137086
0.02	+6.293481	−28.211485	0.054	+5.544940	−17.770308
0.03	+6.035243	−23.770438	0.056	+5.509752	−17.419527
0.04	+5.813417	−20.760417	0.058	+5.475252	−17.083578

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.060	+5.441410	−16.761411	0.144	+4.401643	− 9.374993
0.062	+5.408198	−16.452077	0.146	+4.382993	− 9.274725
0.064	+5.375594	−16.154726	0.148	+4.364542	− 9.176358
0.066	+5.343573	−15.868581	0.150	+4.346287	− 9.079836
0.068	+5.312113	−15.592941	0.160	+4.257807	− 8.623075
0.070	+5.281194	−15.327167	0.170	+4.173696	− 8.205179
0.072	+5.250797	−15.070671	0.180	+4.093590	− 7.821216
0.074	+5.220905	−14.822918	0.190	+4.017171	− 7.467088
0.076	+5.191501	−14.583416	0.200	+3.944160	− 7.139361
0.078	+5.162566	−14.351713	0.210	+3.874306	− 6.835131
0.080	+5.134089	−14.127388	0.220	+3.807387	− 6.551918
0.082	+5.106052	−13.910058	0.230	+3.743205	− 6.287593
0.084	+5.078444	−13.699361	0.240	+3.681579	− 6.040318
0.086	+5.051250	−13.494968	0.250	+3.622347	− 5.808491
0.088	+5.024460	−13.296571	0.260	+3.565362	− 5.590710
0.090	+4.998061	−13.103881	0.270	+3.510490	− 5.385747
0.092	+4.972041	−12.916632	0.280	+3.457608	− 5.192514
0.094	+4.946391	−12.734573	0.290	+3.406604	− 5.010047
0.096	+4.921099	−12.557470	0.300	+3.357374	− 4.837487
0.098	+4.896157	−12.385104	0.325	+3.241457	− 4.444589
0.100	+4.871556	−12.217270	0.350	+3.134751	− 4.099007
0.102	+4.847285	−12.053776	0.375	+3.036177	− 3.792961
0.104	+4.823337	−11.894440	0.400	+2.944825	− 3.520314
0.106	+4.799704	−11.739091	0.425	+2.859924	− 3.276149
0.108	+4.776378	−11.587570	0.450	+2.780814	− 3.056481
0.110	+4.753352	−11.439723	0.475	+2.706923	− 2.858040
0.112	+4.730618	−11.295408	0.500	+2.637758	− 2.678114
0.114	+4.708168	−11.154490	0.525	+2.572883	− 2.514441
0.116	+4.685997	−11.016841	0.550	+2.511916	− 2.365108
0.118	+4.664098	−10.882339	0.575	+2.454521	− 2.228498
0.120	+4.642465	−10.750871	0.600	+2.400397	− 2.103222
0.122	+4.621092	−10.622326	0.625	+2.349275	− 1.988089
0.124	+4.599974	−10.496601	0.650	+2.300917	− 1.882070
0.126	+4.579104	−10.373598	0.675	+2.255103	− 1.784268
0.128	+4.558477	−10.253223	0.700	+2.211641	− 1.693901
0.130	+4.538090	−10.135388	0.750	+2.131076	− 1.532806
0.132	+4.517934	−10.020008	0.800	+2.057986	− 1.394205
0.134	+4.498008	− 9.907001	0.850	+1.991341	− 1.274491
0.136	+4.478305	− 9.796291	0.900	+1.930270	− 1.170787
0.138	+4.458821	− 9.687804	0.950	+1.874034	− 1.080768
0.140	+4.439552	− 9.581470	1.000	+1.821997	− 1.002539
0.142	+4.420495	− 9.477221			
		$\varkappa = 1$			
0	+9.223177	−∞	0.062	+6.861733	−22.733834
0.01	+8.537488	−50.941125	0.064	+6.816689	−22.313237
0.02	+8.092535	−39.474702	0.066	+6.772470	−21.908662
0.03	+7.731907	−33.131908	0.068	+6.729044	−21.519101
0.04	+7.423209	−28.845587	0.070	+6.686384	−21.143634
0.05	+7.151359	−25.665138	0.072	+6.644462	−20.781419
0.052	+7.100576	−25.120414	0.074	+6.603250	−20.431693
0.054	+7.050860	−24.600480	0.076	+6.562727	−20.093744
0.056	+7.002160	−24.103453	0.078	+6.522868	−19.766922
0.058	+6.954432	−23.627651	0.080	+6.483652	−19.450631
0.060	+6.907637	−23.171563	0.082	+6.445059	−19.144314

(Table I, Cont.)

ϱ	v	v'	ϱ	v	v'
0.084	+6.407068	−18.847461	0.180	+5.060945	−10.619357
0.086	+6.369663	−18.559594	0.190	+4.957242	−10.127782
0.088	+6.332825	−18.280271	0.200	+4.858265	− 9.673380
0.090	+6.296536	−18.009081	0.210	+4.763664	− 9.252028
0.092	+6.260783	−17.745643	0.220	+4.673126	− 8.860207
0.094	+6.225549	−17.489594	0.230	+4.586371	− 8.494900
0.096	+6.190820	−17.240603	0.240	+4.503148	− 8.153496
0.098	+6.156582	−16.998356	0.250	+4.423229	− 7.833732
0.100	+6.122822	−16.762557	0.260	+4.346408	− 7.533626
0.102	+6.089527	−16.532931	0.270	+4.272497	− 7.251438
0.104	+6.056686	−16.309221	0.280	+4.201324	− 6.985632
0.106	+6.024286	−16.091180	0.290	+4.132734	− 6.734847
0.108	+5.992317	−15.878578	0.300	+4.066581	− 6.497873
0.110	+5.960769	−15.671202	0.325	+3.911020	− 5.959030
0.112	+5.929629	−15.468844	0.350	+3.768083	− 5.485927
0.114	+5.898890	−15.271314	0.375	+3.636268	− 5.067628
0.116	+5.868541	−15.078424	0.400	+3.514317	− 4.695518
0.118	+5.838573	−14.890003	0.425	+3.401165	− 4.362710
0.120	+5.808977	−14.705889	0.450	+3.295900	− 4.063628
0.122	+5.779747	−14.525924	0.475	+3.197741	− 3.793705
0.124	+5.750872	−14.349960	0.500	+3.106004	− 3.549160
0.126	+5.722344	−14.177858	0.525	+3.020097	− 3.326843
0.128	+5.694158	−14.009483	0.550	+2.939499	− 3.124097
0.130	+5.666304	−13.844708	0.575	+2.863749	− 2.938673
0.132	+5.638777	−13.683414	0.600	+2.792437	− 2.768650
0.134	+5.611568	−13.525481	0.625	+2.725202	− 2.612379
0.136	+5.584673	−13.370804	0.650	+2.661716	− 2.468437
0.138	+5.558084	−13.219275	0.675	+2.601687	− 2.335586
0.140	+5.531794	−13.070797	0.700	+2.544853	− 2.212750
0.142	+5.505798	−12.925270	0.750	+2.439835	− 1.993456
0.144	+5.480091	−12.782604	0.800	+2.345006	− 1.804267
0.146	+5.454666	−12.642712	0.850	+2.258989	− 1.640248
0.148	+5.429518	−12.505509	0.900	+2.180628	− 1.497472
0.150	+5.404642	−12.370915	0.950	+2.108941	− 1.372782
0.160	+5.284163	−11.734499	1.000	+2.043097	− 1.263600
0.170	+5.169769	−11.152986			

7. Computation of the kernel $\varphi(\alpha, \lambda)$

From now on we restrict the variables α and ϱ by introducing the condition $2\alpha\varrho = \lambda$ with $\lambda = \varkappa r$ and $\varkappa = Me^2/\hbar^2$ (cf. eq. (15)–(17)). The physically interesting values of λ lie in the range 0.02–0.1 and according to this we choose $\lambda = n \cdot 0.012$; $n = 1, 2, \ldots 10$.

The function $\varphi(\alpha, \lambda)$ has been computed from [25] and table I; for small values of ϱ checking has been performed using the formula of YOST, WHEELER and BREIT ([28], p. 53; [31], [32] and [33]). This formula can be written:

$$\left\{\begin{aligned}
v &= \frac{e^{2\pi\alpha}-1}{\pi}\left[\log 2\varrho + 2\gamma - 1 + \text{R. P.}\,\frac{\Gamma'(i\alpha)}{\Gamma(i\alpha)}\right] u + 2\alpha\sqrt{\frac{e^{2\pi\alpha}-1}{2\pi\alpha}}\,\cdot \\
&\quad \cdot\left\{\cos\varrho\left[\frac{1}{2\alpha} - A_1\varrho + A_3\varrho^3 - \cdots\right] - \sin\varrho\,[A_2\varrho^2 - A_4\varrho^4 + \cdots]\right\} \\
v' &= \frac{e^{2\pi\alpha}-1}{\pi}\left[\log 2\varrho + 2\gamma - 1 + \text{R. P.}\,\frac{\Gamma'(i\alpha)}{\Gamma(i\alpha)}\right] u' + \\
&\quad + \frac{e^{2\pi\alpha}-1}{\pi\varrho}\cdot u + 2\alpha\cdot\sqrt{\frac{e^{2\pi\alpha}-1}{2\pi\alpha}}\cdot\left\{\cos\varrho\,[-B_0 + B_2\varrho^2 - B_4\varrho^4 + \cdots] - \right. \\
&\quad \left. - \sin\varrho\left[\frac{1}{2\alpha} + B_1\varrho - B_3\varrho^3 + \cdots\right]\right\}.
\end{aligned}\right.$$

Here the coefficients A_i and B_i are polynomials in λ:

$$A_1 = \frac{3}{4}\lambda + \frac{7}{36}\lambda^2 + \frac{35}{1728}\lambda^3 + \frac{101}{86400}\lambda^4 + \frac{7}{162000}\lambda^5 + \cdots$$

$$A_2 = \frac{1}{2} + \frac{5}{6}\lambda + \frac{29}{144}\lambda^2 + \frac{89}{4320}\lambda^3 + \frac{51}{43200}\lambda^4 + \cdots$$

$$A_3 = \frac{5}{9} + \frac{241}{432}\lambda + \frac{527}{4320}\lambda^2 + \frac{193}{16200}\lambda^3 + \cdots$$

$$A_4 = \frac{13}{36} + \frac{59}{216}\lambda + \frac{117}{2160}\lambda^2 + \cdots$$

$$A_5 = \frac{77}{450} + \frac{4297}{40500}\lambda + \cdots$$

$$A_6 = \frac{87}{1350} + \frac{1387}{40500}\lambda + \cdots$$

$$A_7 = \frac{223}{11025} + \cdots$$

$$B_0 = \frac{3}{2}\lambda + \frac{7}{12}\lambda^2 + \frac{35}{432}\lambda^3 + \frac{101}{17280}\lambda^4 + \frac{7}{27000}\lambda^5 + \cdots$$

$$B_1 = 1 + \frac{7}{4}\lambda + \frac{11}{18}\lambda^2 + \frac{143}{1728}\lambda^3 + \frac{511}{86400}\lambda^4 + \cdots$$

$$B_2 = \frac{7}{6} + \frac{151}{108}\lambda + \frac{353}{864}\lambda^2 + \frac{1099}{21600}\lambda^3 + \cdots$$

$$B_3 = \frac{8}{9} + \frac{349}{432}\lambda + \frac{877}{4320}\lambda^2 + \cdots$$

$$B_4 = \frac{89}{189} + \frac{3271}{9000}\lambda + \cdots$$

$$B_5 = \frac{97}{450} + \frac{451}{3375}\lambda + \cdots$$

$$B_6 = \frac{27}{350} + \cdots$$

The double-polynomials within brackets have also been computed on the BARK. For not too large values of λ and ϱ 5–6 digits accuracy was easily obtained.

At last we also want to examine the behaviour of $\varphi(\alpha, \lambda)$ for small and for large values of α. In [11] it was proved that for $l = 0$:

$$\lim_{\varrho \to 0} (u' v + u v') = 1$$

under the supposition that α has a fixed value. This does not imply, however, that $\lim_{\alpha \to \infty} \varphi(\alpha, \lambda)$ should also be unity since α and ϱ are not independent of each other. For the computation we need the series expansion of u and u':

$$\left.\begin{aligned} u &= \sqrt{\frac{2\pi\alpha}{e^{2\pi\alpha} - 1}}\,(\varrho + \beta_2 \varrho^2 + \cdots) \\ u' &= \sqrt{\frac{2\pi\alpha}{e^{2\pi\alpha} - 1}}\,(1 + \beta_1' \varrho + \cdots) \end{aligned}\right\} \tag{23}$$

with

$$\left.\begin{aligned} &k(k+1)\beta_{k+1} - 2\alpha\beta_k + \beta_{k-1} = 0 \\ &\beta_{k-1}' = k\beta_k . \end{aligned}\right\} \tag{24}$$

From (21) and (23) we obtain after some calculations:

$$\begin{aligned} \lim_{\alpha \to \infty} \varphi(\alpha, \lambda) = 1 + 4\gamma\lambda + (6\gamma - 4)\lambda^2 + \frac{20\gamma - 21}{6}\lambda^3 + \frac{105\gamma - 139}{108}\lambda^4 + \cdots \\ + \log\lambda \cdot \left[2\lambda + 3\lambda^2 + \frac{5}{3}\lambda^3 + \frac{35}{72}\lambda^4 + \cdots\right]. \end{aligned} \tag{25}$$

On the other hand for small values of α we have

$$\varphi(\alpha, \lambda)_{\alpha \to 0} \sim \cos\left(\frac{\lambda}{\alpha} - 2\alpha \log\frac{\lambda}{\alpha} + 2 \arg \Gamma(i\alpha + 1)\right). \tag{26}$$

8. Table II

This table contains the function $\varphi(\alpha, \lambda)$. The error will probably not exceed one unit in the fifth place. For smaller values of α formula (26) can be used instead with at least 4 digits accuracy.

$\alpha \diagdown \lambda$	0.012	0.024	0.036	0.048	0.060	$\alpha \diagdown \lambda$	0.072	0.084	0.096	0.108	0.120
0.01	0.376319	0.717522	−0.912560	0.044724	0.945872	0.01	0.648018	−0.472332	−0.992996	−0.251946	0.809440
0.015	0.703942	0.001918	−0.707408	−0.999982	−0.696981	0.015	0.023332	0.730255	0.999041	0.667210	−0.066329
0.02	0.826336	0.389883	−0.182056	−0.697180	−0.978379	0.02	−0.927059	−0.558588	0.001954	0.562349	0.929478
0.025	0.882766	0.594294	0.176057	−0.284474	−0.686887	0.025	−0.941922	−0.991652	−0.823861	−0.474246	−0.019400
0.03	0.912450	0.710412	0.403043	0.032425	−0.346613	0.03	−0.676401	−0.905905	−0.998980	−0.940250	−0.737744
0.035	0.929444	0.781024	0.549926	0.257088	−0.066859	0.035	−0.386459	−0.665928	−0.873343	−0.984863	−0.987063
0.04	0.939720	0.826294	0.648212	0.415797	0.146600	0.04	−0.137503	−0.412638	−0.655216	−0.844106	−0.962580
0.06	0.954537	0.902539	0.825297	0.721096	0.591535	0.06	0.440061	0.271437	0.091383	−0.093720	−0.277126
0.08	0.956245	0.922665	0.879544	0.823367	0.753130	0.08	0.668954	0.571675	0.462651	0.343654	0.216790
0.10	0.954905	0.927708	0.898476	0.863297	0.820550	0.10	0.769537	0.710062	0.642269	0.566562	0.483549
0.12	0.952751	0.927555	0.904417	0.879316	0.850458	0.12	0.816892	0.778102	0.733837	0.684033	0.628770
0.16	0.948199	0.922493	0.902857	0.885225	0.867732	0.16	0.849316	0.829304	0.807247	0.782839	0.755867
0.20	0.944145	0.916182	0.896070	0.879745	0.865336	0.20	0.851770	0.838356	0.824612	0.810194	0.794841
0.25	0.939938	0.908850	0.886694	0.869404	0.855106	0.25	0.842734	0.831596	0.821213	0.811234	0.801392
0.3	0.936541	0.902605	0.878139	0.859077	0.843549	0.3	0.830489	0.819209	0.809233	0.800212	0.791881
0.4	0.931538	0.893081	0.864566	0.841923	0.823279	0.4	0.807572	0.794113	0.782433	0.772185	0.763108
0.5	0.928173	0.886537	0.855019	0.829543	0.808237	0.5	0.790037	0.774255	0.760418	0.748186	0.737295
0.6	0.925877	0.882021	0.848362	0.820817	0.797518	0.6	0.777391	0.759754	0.744130	0.730178	0.717639
0.7	0.924281	0.878882	0.843707	0.814688	0.789943	0.7	0.768405	0.749391	0.732421	0.717153	0.703323
0.8	0.923163	0.876670	0.840425	0.810353	0.784575	0.8	0.762025	0.742011	0.724053	0.707815	0.693034
0.9	0.922375	0.875100	0.838088	0.808806	0.780738	0.9	0.757455	0.736716	0.718051	0.701106	0.685629
1.0	0.921845	0.873932	0.836379	0.805013	0.777951	1.0	0.754179	0.732906	0.713701	0.696236	0.680281
∞	0.919555	0.869525	0.829818	0.796346	0.767227	∞	0.741386	0.718131	0.696986	0.677602	0.659717

9. Determination of the first approximation $V(r)$

With the aid of Table II and the experimentally known function $\sin\delta(\alpha)$ it is possible, at least in principle, to determine the first approximation of the potential using equation [16]. As can be expected there are some serious difficulties to be overcome. First: we know very little about the ^1S phase shift for energies beyond 30 MeV, although experiments have been performed even for energies as high as 345 MeV. Second: the factor α^{-3} in the integrand is apt to cause trouble at the numerical integration. At first we will procceed as follows. For $\alpha > 0.02$ (corresponding to $\varepsilon < 30$ MeV) we will integrate numerically straight forward. For small values of α we know theoretically that $\sin\delta$ tends to zero (cf. [20]). In order to secure convergence we must cleim

$$\sin\delta \sim \alpha^{1+p}, \quad p > 0,$$

and the principle of least computation suggests

$$\sin\delta = c\,\alpha^2 \text{ (i.e. } \sim \varepsilon^{-1}) \cdot \text{for } \alpha < 0.02.$$

As previously mentioned $\varphi(\alpha, \lambda) \sim \cos\frac{\lambda}{\alpha}$ for small valueus of α and thus we have to tabulate

$$c \cdot \int\limits_0^{\alpha_0} \frac{d\alpha}{\alpha} \cos\frac{\lambda}{\alpha} = c \int\limits_{\lambda/\alpha_0}^{\infty} \frac{\cos x}{x}\, dx = -c \cdot Ci\left(\frac{\lambda}{\alpha_0}\right)$$

for $\alpha_0 = 0.02$ and for different values of λ.

As a matter of fact we can even use the better approximation

$$\varphi(\alpha, \lambda) \sim \cos\left(\frac{\lambda}{\alpha} - 2\,\alpha \log\frac{\lambda}{\alpha} + 2 \arg\Gamma(1 + i\alpha)\right) \cdot$$

Subtracting $\cos\frac{\lambda}{\alpha}$ we obtain:

$$\varphi(\alpha, \lambda) - \cos\frac{\lambda}{\alpha} \sim \sin\frac{\lambda}{\alpha} \cdot \sin\left(2\,\alpha \log\frac{\lambda}{\alpha} - 2 \arg\Gamma(1 + i\alpha)\right)$$

and using

$$\arg\Gamma(1 + i\alpha) = -\gamma\alpha + \frac{\zeta(3)}{3}\alpha^3 - \frac{\zeta(5)}{5}\alpha^5 + \cdots$$

we get:

$$\varphi(\alpha, \lambda) - \cos\frac{\lambda}{\alpha} \sim 2\,\alpha\left(\gamma + \log\frac{\lambda}{\alpha}\right) \sin\frac{\lambda}{\alpha} \cdot \tag{27}$$

Hence we have the correction term

$$2c\int\limits_0^{\alpha_0}\left(\gamma+\log\frac{\lambda}{\alpha}\right)\sin\frac{\lambda}{\alpha}\,d\alpha = 2c\lambda\int\limits_{\lambda/\alpha_0}^{\infty}(\gamma+\log x)\frac{\sin x}{x^2}\,dx =$$

$$= 2c\alpha_0(1+\gamma+\log\beta)(\sin\beta-\beta\,Ci(\beta)) - 2c\lambda\int\limits_{\beta}^{\infty}\frac{Ci(x)}{x}\,dx$$

where $\beta=\lambda/\alpha_0$. The function $Ci(x)$ was taken from standard tables and the integral $\int\limits_{\beta}^{\infty} Ci(x)/x\cdot dx$ could be computed from the semi-convergent series and numerical integration. Our first approximation can now be written:

$$V_1(r) = -\frac{2}{\pi}\varkappa^2 f(\lambda) \quad (\lambda=\varkappa r),$$

where:

$$f(\lambda)=\int\limits_0^{\infty}\frac{\varphi(\alpha,\lambda)\sin\delta(\alpha)}{\alpha^3}\,d\alpha \backsimeq \int\limits_{\alpha_0}^{\infty}\frac{\varphi(\alpha,\lambda)\sin\delta(\alpha)}{\alpha^3}\,d\alpha -$$

$$-c\cdot\left\{Ci(\beta)-2\alpha_0(1+\gamma+\log\beta)(\sin\beta-\beta\,Ci(\beta))+2\lambda\int\limits_{\beta}^{\infty}\frac{Ci(x)}{x}\,dx\right\}. \tag{28}$$

The function $f(\lambda)$ appears in Table III a.

Table III a

λ	0.012	0.024	0.036	0.048	0.060
$f(\lambda)$	942	−132	−541	−644	−583
λ	0.072	0.084	0.096	0.108	0.120
$f(\lambda)$	−441	−277	−133	−42	−2

It turns out that $f(\lambda)$ is extremely sensitive to changes in $\sin\delta(\alpha)$, and we can at once draw the important but hardly surprising conclusion that accurate high energy scattering experiments are necessary for determining the potential function.

In this situation we observe that there exists a very natural way of avoiding the difficulties mentioned before, namely making use of a cut-off convention. For simple physical reasons it is clear that the potential for small values of r and the large energy scattering should be connected with each other to a very high extent. Thus it seems natural to choose the cut-off radius proportional to λ, and we will try explicitly with $\lambda/2$ and λ. With our choice of $\lambda = n\cdot 0.012$ the smallest value of α will be 0.006, equivalent to $\varepsilon = 345$ MeV. Knowing practically nothing of the phase shift for energies between 30 and 345 MeV we will assume $\sin\delta=$ const. ($=0.79$) for $0.006<\alpha<0.02$.

Then we obtain the following integral:

$$I = \int_a^b \frac{\varphi(\alpha, \lambda)}{\alpha^3} d\alpha$$

and using (27) this transforms to

$$I = \frac{1}{\lambda^2} \int_{\lambda/b}^{\lambda/a} x \cdot dx \cdot \left\{ \cos x + \frac{2\lambda}{x} (\gamma + \log x) \sin x \right\} =$$

$$= \frac{1}{\lambda^2} \cdot \Big[x \sin x + \cos x + 2\lambda (Ci(x) - (\gamma + \log x) \cos x) \Big]_{\lambda/b}^{\lambda/a}.$$

For convenience we introduce the notations:

$$g(\lambda) = \int_{\lambda/2}^{\infty} \frac{\varphi(\alpha, \lambda) \sin \delta(\alpha)}{\alpha^3} d\alpha$$

and

$$h(\lambda) = \int_{\lambda}^{\infty} \frac{\varphi(\alpha, \lambda) \sin \delta(\alpha)}{\alpha^3} d\alpha.$$

The functions $g(\lambda)$ and $h(\lambda)$ are given in Table III b and Fig. 2 below.

Table III b

λ	0.012	0.024	0.036	0.048	0.060
$g(\lambda)$	2141	525	239	145	86.7
$h(\lambda)$	1905	435	205	103	63.7
λ	0.072	0.084	0.096	0.108	0.120
$g(\lambda)$	51.5	34.4	27.9	20.3	15.3
$h(\lambda)$	36.2	22.5	14.5	9.7	6.6

It turns out that $g(\lambda)$ and $h(\lambda)$ can be represented very nicely by Yukawa functions, and we easily find the range r_0 and the strength J. Denoting the cut-off radius by a we get the following results.

Table IV

a	r_0 (cm)	J (MeV)
$\lambda/2$	$1.21 \cdot 10^{-13}$	18.2
λ	$1.00 \cdot 10^{-13}$	22.0

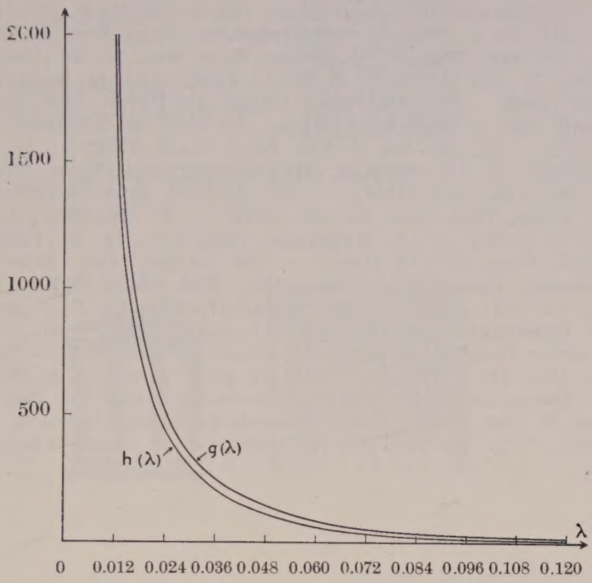

Fig. 2.

These values should be compared e.g. with the values in [30], p. 153, especially $r_0 = 1.37 \cdot 10^{-13}$ and $J = 32.6$, corresponding to the meson mass 278. Remembering that we have treated the first approximation only we can hardly expect any better agreement.

At last it should be observed that when better experimental values for $\sin \delta(\alpha)$ are available, the corresponding first approximation of the potential function can be computed very rapidly with the aid of Table II.

Acknowledgements

The author is indebted to Dr. A. N. Lowan, Math. Tables Project, New York, for placing at his disposal the manuscript of the tables on regular Coulomb wave functions. For valuable assistance at the numerical calculations I want to extend my sincere thanks to Mr. S. Sköld and Mrs. S. Wollstad. Further my thanks are due to Matematikmaskinnämnden for giving me an opportunity to use the automatic relay machine BARK for computation of the irregular Coulomb wave functions. For supervising the whole running of the problem and for close collaboration during the planning and coding I want to express my gratitude to fil. kand. G. Kjellberg, civiling. G. Neovius and civiling. O. Karlqvist. Lastly I am also indebted to Atomkommittén for financial support.

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Tryckt den 26 februari 1951

Uppsala 1951. Almqvist & Wiksells Boktryckeri AB